LECTURE 16: GRAVITY

Let's review Newtonian Gravity, for M at origin, the mass m has

$$\vec{F} = -\frac{GmM}{r^2}\hat{r} = m\vec{a} \longrightarrow \vec{a} = -\frac{GM}{r^2}\hat{r}$$

The gravitational potential Φ is related by $\frac{\vec{F}}{m} = -\nabla\Phi$

Then $-\frac{GM}{r^2}\hat{r} = -\nabla\Phi \Rightarrow \frac{\partial\Phi}{\partial r} = \frac{GM}{r^2} \therefore \Phi = \Phi_0 - \frac{GM}{r}$

Near surface of Earth, $r = R_E + h$ where h = altitude then,

$$\Phi(R_E+h) = \Phi(R_E) + \Phi'(R_E)h + \frac{1}{2}\Phi''(R_E)h^2 + \dots$$

$$\Phi(R_E+h) = \frac{GM}{R_E^2}h - \frac{2GM}{R_E^3}h^2 + \dots \qquad (\text{setting } \Phi(R_E) = 0)$$

Here $G = 6.674\times10^{-11}\ \frac{Nm^2}{kg^2}$ and $R_E = 6.3781\times10^6\,m$

and $M = 5.972\times10^{24}\,kg$ then calculate $\frac{GM}{R_E^2} = 9.798\,\frac{m}{s^2} \approx 9.8\frac{m}{s^2} = $ "g"

Therefore, $\Phi(R_E+h) \cong gh$ and PE for m is $PE \approx mgh$.

$$\int_{S_R}\vec{a}\cdot d\vec{S} = -\int_{S_R}\left(\frac{GM}{r^2}\hat{r}\right)\cdot\left(R^2 d\Omega\,\hat{r}\right) = -\int_{S_R}\frac{GM}{R^2}\cdot R^2 d\Omega = -4\pi GM = -4\pi G\int_{B_R}\rho\, d^3V$$

$$= -\int_{B_R}\underbrace{\nabla\cdot\left(\frac{GM}{r^2}\hat{r}\right)}_{4\pi\delta(\vec{r}) = \nabla\cdot\left(\frac{\hat{r}}{r^2}\right)}d^3V = -\int_{B_R}(\nabla\cdot\nabla\Phi)\,d^3V \Rightarrow \boxed{\nabla^2\Phi = 4\pi G\rho}$$

(I think, $\rho = M\delta(\vec{r})$ for point mass at origin.)

$$\Phi = -\frac{GM}{r}$$

$$\nabla\Phi = -\frac{\partial}{\partial r}\left(\frac{GM}{r}\right)\hat{r} = \frac{GM}{r^2}\hat{r}$$

DIRAC DELTA IN 3D

$$\nabla^2\Phi = \nabla\cdot\nabla\Phi = GM\,\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) = GM\left(4\pi\,\delta(\vec{r})\right)$$

$\therefore\ \nabla^2\Phi = 4\pi G\rho$ where $\rho = M\delta(\vec{r})$ for point mass at $\vec{r}=0$.

Electrostatics?

$\vec{E} = \frac{\vec{F}}{q}$ and $\vec{E} = -\nabla\Phi$

Gauss' Law $\nabla\cdot\vec{E} = \rho/\epsilon_0$ where $\rho = \frac{dQ}{dV} =$ charge density

$$\nabla\cdot(-\nabla\Phi) = \rho/\epsilon_0$$

$\boxed{\nabla^2\Phi = -\rho/\epsilon_0}$ Poisson's Eq: of Electrostatics.

Coulomb's Law $\vec{F} = \frac{qQ}{4\pi\epsilon_0}\frac{\hat{r}}{r^2} \hookrightarrow \vec{E} = \frac{\vec{F}}{q} = kQ\frac{\hat{r}}{r^2}$ with $k = \frac{1}{4\pi\epsilon_0}$

So Poisson's Eq: is also $\boxed{\nabla^2\Phi = -4\pi k\rho}$

(difference in sign due to fact gravity is <u>attractive</u>)

How to find the law of gravity... minimal-coupling principle

1.) Take a law of physics, valid in an inertial coordinate system in flat spacetime
2.) Write the law in coordinate invariant form (with tensors)
3.) $\underbrace{\text{Assert the resulting law holds in curved spacetime}}_{\text{by replacing } \eta_{\mu\nu} \text{ with } g_{\mu\nu} \text{ and } \partial_\mu \text{ with } \nabla_\mu}$

<u>Example</u>: free-falling, unaccelerated particle

$$\frac{d^2x^\mu}{d\lambda^2} = 0 \longrightarrow \frac{d}{d\lambda}\left(\frac{d}{d\lambda}(x^\mu)\right) = 0$$

$$\frac{dx^\nu}{d\lambda}\,\partial_\nu \frac{dx^\mu}{d\lambda} = 0$$

chain-rule, now stated in manifestly tensorial fashion.

Replace ∂_ν with $\nabla_\nu = \partial_\nu + \Gamma_{\cdot\cdot}^{\ \ ?}$

$$\frac{dx^\nu}{d\lambda}\nabla_\nu \frac{dx^\mu}{d\lambda} = \frac{dx^\nu}{d\lambda}\left[\partial_\nu \frac{dx^\mu}{d\lambda} + \Gamma^\mu_{\nu\lambda}\frac{dx^\lambda}{d\lambda}\right]$$

$$\underline{0 = \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda}}$$

(In G.R. free particles follow geodesics)

Motivating Equations of GR from Newtonian Limit

Moving slowly $\frac{dx^i}{d\tau} \ll \frac{dt}{d\tau}$, weak field $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$ and the gravitational field is static (no time dependence for $g_{\mu\nu}$)

In this context, study geodesic eq.:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2 = 0 \qquad \left(\text{other terms small in comparison } \frac{dx^i}{d\tau} \ll \frac{dt}{d\tau}\right)$$

$$\Gamma^\mu_{00} = \frac{1}{2} g^{\mu\lambda}\left(\partial_0 g_{\lambda 0} + \partial_0 g_{0\lambda} - \partial_\lambda g_{00}\right) = -\frac{1}{2} g^{\mu\lambda}\partial_\lambda g_{00}$$

Since $g^{\mu\nu} g_{\nu\sigma} = \delta^\mu_\sigma$ we can approximate $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$ as

$$(\eta_{\mu\nu} + h_{\mu\nu})(\eta^{\nu\sigma} - h^{\nu\sigma}) = \eta_{\mu\nu}\eta^{\nu\sigma} \underbrace{- \eta_{\mu\nu}h^{\nu\sigma} + h_{\mu\nu}\eta^{\nu\sigma}}_{\text{these cancel to 1st order in } h \text{ as}} - h_{\mu\nu}h^{\nu\sigma} \approx \delta^\sigma_\mu$$

$$h^{\nu\sigma} = \eta^{\nu\delta}\eta^{\sigma\rho}h_{\delta\rho}$$

Continuing,

$$\frac{d^2x^\mu}{d\tau^2} = \frac{1}{2} g^{\mu\lambda}\partial_\lambda g_{00}\left(\frac{dt}{d\tau}\right)^2$$

$$\rightarrow \frac{d^2t}{d\tau^2} = 0 \quad \text{since } \partial_0 h_{00} = 0$$

Also,

$$\frac{d^2x^i}{d\tau^2} = \frac{1}{2} g^{ij}\partial_j (g_{00})\cancel{\left(\frac{dt}{d\tau}\right)^2} = \frac{1}{2}(\partial_i h_{00})\left(\frac{dt}{d\tau}\right)^2$$

$$\frac{1}{\left(\frac{dt}{d\tau}\right)^2}\frac{d^2x^i}{d\tau^2} = \frac{1}{2}\partial_i h_{00}$$

$$* \Rightarrow \underline{\frac{d^2x^i}{dt^2} = \frac{1}{2}\partial_i h_{00}}.$$

(*) $$\frac{d^2x^i}{dt^2} = \frac{d}{dt}\left[\frac{dx^i}{dt}\right] = \underbrace{\frac{d}{dt}\left[\frac{d\tau}{dt}\frac{dx^i}{d\tau}\right] = \frac{d\tau}{dt}\frac{d}{dt}\left[\frac{dx^i}{d\tau}\right]}_{\text{using } \frac{d^2t}{d\tau^2}=0} = \frac{d\tau}{dt}\frac{d\tau}{dt}\frac{d^2x^i}{d\tau^2}$$

We've shown,

$$\frac{d^2x^i}{dt^2} = \frac{1}{2}\partial_i h_{00} \quad \text{compare to} \quad \vec{a} = -\nabla\Phi \Rightarrow \underline{h_{00} = -2\Phi}.$$

$$\therefore \; g_{00} = \eta_{00} + h_{00}$$

$$\Rightarrow \underline{g_{00} = -1 - 2\Phi}.$$

Einstein's Equation

$$R_{\mu\nu} = \kappa T_{\mu\nu} \quad (?)$$

well, since $\nabla^\mu T_{\mu\nu} = 0$ from conservation of momentum/energy we'd have $\nabla^\mu R_{\mu\nu} = 0$ which is too limiting... note that $\nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R$ thus $R = \kappa g^{\mu\nu} T_{\mu\nu} = \kappa T \Rightarrow \nabla_\mu T = 0 \Rightarrow T$ constant throughout spacetime. (Physically absurd)

Instead, try $G_{\mu\nu} = \kappa T_{\mu\nu}$ where $\underbrace{G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}}_{\text{symmetric, conserved tensor}}$

Next, we need to fix κ by examining the Newtonian limit, 6

$$G_{\mu\nu} = \kappa T_{\mu\nu}$$

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu} \longrightarrow g^{\mu\nu} R_{\mu\nu} - \tfrac{1}{2} R g^{\mu\nu} g_{\mu\nu} = \kappa g^{\mu\nu} T_{\mu\nu}$$

$$R - \tfrac{1}{2} R (4) = \kappa T$$

$$\therefore \; -R = \kappa T \rightarrow \underline{R = -\kappa T}_{*}$$

From $*$,

$$R_{\mu\nu} = \tfrac{1}{2} R g_{\mu\nu} + \kappa T_{\mu\nu}$$

$$\boxed{R_{\mu\nu} = \kappa \left(T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T \right)}$$

Remark: in the absence of matter/energy $T_{\mu\nu} = 0$ and $T = 0$ thus $R_{\mu\nu} = 0$, so space is flat away from matter/energy. (here I'm not taking cosmological considerations into account...)

Problem: determine value of κ from examining weak field Newtonian limit.
Consider perfect-fluid source of energy-momentum where

$$T_{\mu\nu} = (\rho + p) U_\mu U_\nu + p g_{\mu\nu}$$

U^μ fluid 4-velocity
ρ rest-frame energy density
p rest-frame momentum density

we can neglect pressure for Newtonian limit, we're just looking at dust

$$T_{\mu\nu} = \rho U_\mu U_\nu$$

in rest frame we have $U^\mu = (U^0, 0, 0, 0)$ and $g_{\mu\nu} U^\mu U^\nu = -1$

as before
$$\left. \begin{aligned} g_{00} &= -1 + h_{00} \\ g^{00} &= -1 - h_{00} \end{aligned} \right\} \quad g_{00} U^0 U^0 = (-1 + h_{00}) U^0 U^0 = -1$$

ok, so Carrol claims $U^0 = 1$ & $U_0 = -1$
in for the approx. considered.

continuing, since $U_0 = -1$ and $U_i = 0$ for $i = 1,2,3$

(ALL OF THIS IS NEWTONIAN LIMIT)

$$T_{\mu\nu} = \rho U_\mu U_\nu = \begin{cases} \rho & \mu = \nu = 0 \\ 0 & \text{otherwise} \end{cases}$$

in the Newtonian limit $T_{00} = \rho$ is much larger than other components.

$$T = g^{\mu\nu} T_{\mu\nu} = g^{00} T_{00} = (-1 - h_{00})\rho \cong -\rho$$

Then we obtain: (here h_{00} small in contrast to rest energy ρ)

$$R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T\right) \Rightarrow R_{00} = \kappa\left(T_{00} - \tfrac{1}{2} g_{00} T\right)$$

$$= \kappa\left(\rho - \tfrac{1}{2}(-1)(-\rho)\right)$$

$$= \tfrac{1}{2}\kappa\rho \quad *$$

Therefore, to connect back to the metric recall,

$$R_{00} = R^\lambda{}_{0\lambda 0} = R^i{}_{0i0} \quad \text{since } R^0{}_{000} = 0$$

Consider,

$$R^i{}_{0j0} = \partial_j \Gamma^i_{00} - \underbrace{\partial_0 \Gamma^i_{j0}}_{\text{ignore, looking at static limit}} + \underbrace{\Gamma^i_{j\lambda}\Gamma^\lambda_{00} - \Gamma^i_{0\lambda}\Gamma^\lambda_{j0}}_{\text{also ignore since these are 2nd order in the metric since } \Gamma \text{ is 1st order in metric}}$$

Hence, all we need to examine is the 1st term)

$$R_{00} = R^i{}_{0i0}$$

$$= \partial_i \Gamma^i_{00}$$

$$= \partial_i\left[\tfrac{1}{2} g^{i\lambda}\left(\partial_0 g_{\lambda 0} + \partial_0 g_{0\lambda} - \partial_\lambda g_{00}\right)\right]$$

$$= -\tfrac{1}{2}\delta^{ij}\partial_i\partial_j h_{00}$$

$$= -\tfrac{1}{2}\nabla^2 h_{00} \quad * \qquad \therefore \quad \nabla^2 h_{00} = -\kappa\rho$$

① We had $\nabla^2\Phi = 4\pi G\rho$

⑤ $h_{00} = -2\Phi$

$$\nabla^2(-2\Phi) = -8\pi G\rho = -\kappa\rho$$

$$\therefore \boxed{\kappa = 8\pi G}$$

Def^n/ Einstein's Equation for general relativity

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

You can derive $R = -8\pi G$ from the trace of the eq^n above, so we can rewrite Einstein's Eq^n as

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right)$$

(we said this before using κ instead of $8\pi G$.)

Lagrangian Formulation

"... Hibert saw... only independent scalar constructed from the metric which is no higher than 2nd order in its derivatives is the Ricci scalar"

Hilbert action $\quad S_H = \int \sqrt{-g}\, R\, d^n x$

study variation of action under small variations of the metric since $g^{\mu\lambda} g_{\lambda\nu} = \delta^{\mu}_{\nu} \Rightarrow \delta g_{\mu\nu} = -g_{\mu\rho} g_{\nu\sigma} \delta g^{\rho\sigma}$ it suffices to study variation w.r.t. inverse metric. Note $R = g^{\mu\nu} R_{\mu\nu}$

$$\delta S_H = \int d^n x \left(\underbrace{\sqrt{-g}\; g^{\mu\nu} \delta R_{\mu\nu}}_{1} + \underbrace{\sqrt{-g}\, R_{\mu\nu}\, \delta g^{\mu\nu}}_{2} + \underbrace{R\, \delta \sqrt{-g}}_{3} \right)$$

Let's go through Carroll's Variational arguments

$$\Gamma^{\rho}_{\nu\mu} \longmapsto \Gamma^{\rho}_{\nu\mu} + \delta\Gamma^{\rho}_{\nu\mu} \longrightarrow \delta\Gamma^{\rho}_{\nu\mu}$$ is difference of connections hence a tensor... so we can ∇_μ

$$\nabla_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) = \partial_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) + \Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\sigma}_{\nu\mu} - \Gamma^{\sigma}_{\lambda\nu}\delta\Gamma^{\rho}_{\sigma\mu} - \Gamma^{\sigma}_{\lambda\mu}\delta\Gamma^{\rho}_{\nu\sigma}$$

$$\longrightarrow \quad \delta R^{\rho}{}_{\mu\lambda\nu} = \nabla_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\rho}_{\lambda\mu})$$

$$\Rightarrow (\delta S)_1 = \int d^n x \sqrt{-g}\, g^{\mu\nu}\left[\nabla_\lambda(\delta\Gamma^{\lambda}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\lambda}_{\lambda\mu})\right]$$

$$= \int d^n x \sqrt{-g}\, \nabla_\sigma\left[g^{\mu\nu}(\delta\Gamma^{\sigma}_{\mu\nu}) - g^{\mu\sigma}(\delta\Gamma^{\lambda}_{\lambda\mu})\right]$$

But, $$\delta\Gamma^{\sigma}_{\mu\nu} = -\frac{1}{2}\left[g_{\lambda\mu}\nabla_\nu(\delta g^{\lambda\sigma}) + g_{\lambda\nu}\nabla_\mu(\delta g^{\lambda\sigma}) - g_{\mu\alpha}g_{\nu\beta}\nabla^\sigma(\delta g^{\alpha\beta})\right]$$

which simplifies $(\delta S)_1$ further,

$$(\delta S)_1 = \int d^n x \sqrt{-g}\, \nabla_\sigma\left[g_{\mu\nu}\nabla^\sigma(\delta g^{\mu\nu}) - \nabla_\lambda(\delta g^{\sigma\lambda})\right]$$

... and... this integrates to zero... so let's turn to $(\delta S)_3$ term

<u>MATRIX</u> <u>IDENTITY</u>: $\ln(\det M) = \text{trace}(\ln M)$ where $\exp(\ln M) = M$ implicitly defines $\ln M$.

$$\Rightarrow \quad \delta(\ln(\det M)) = \delta(\text{trace}(\ln M))$$

$$\frac{1}{\det M}\,\delta(\det M) = \text{Tr}(M^{-1}\delta M)$$

linearity of trace & cyclicity of trace

Let $M = (g_{\mu\nu})$ then $\det(M) = g$ and,

$$\delta g = g\,\text{Tr}(M^{-1}\delta M) = g\,(g^{\mu\nu}\delta g_{\mu\nu}) = -g\,(g_{\mu\nu}\,\delta g^{\mu\nu})$$

$$\delta\sqrt{-g} = \frac{-1}{2\sqrt{-g}}\,\delta g = \frac{g}{2\sqrt{-g}}\,g_{\mu\nu}\,\delta g^{\mu\nu} = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu}$$

In summary,

$$\delta S_H = \int d^n x\,\sqrt{-g}\left[R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}\right]\delta g^{\mu\nu} \quad \Rightarrow \quad \underbrace{R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 0}_{\text{Einstein's Eq}^{2}\text{ in vacuum from}}$$

$$\frac{1}{\sqrt{-g}}\frac{\delta S_H}{\delta g^{\mu\nu}}$$

Next, add matter,

$$S = \frac{1}{16\pi G}\,S_H + S_M$$

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{16\pi G}\left(R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}\right) + \frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} = 0$$

So, <u>define</u> $T_{\mu\nu} = -2\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}$ and obtain $\boxed{R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 8\pi G\,T_{\mu\nu}}$

Scalar Fields and GENERAL RELATIVITY

- The equation of motion for scalar field ϕ subject potential V is

$$\boxed{\Box\phi - \frac{dV}{d\phi} = 0} \quad \text{where} \quad \boxed{\Box = \nabla^\mu \nabla_\mu = g^{\mu\nu} \nabla_\mu \nabla_\nu}$$

(see p. 160 of Carroll)

- The energy/momentum tensor for the scalar field is derived (p. 164) to be

$$\boxed{T^{(\phi)}_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \nabla_\rho \phi \nabla_\sigma \phi - g_{\mu\nu} V(\phi)}$$

- Scalar-tensor theories are modified theories of GR where $S = \underbrace{S_{fR}}_{\text{gravitational}} + \underbrace{S_\lambda}_{\text{pure scalar}} + \underbrace{S_M}_{\text{matter}}$

$$\boxed{G_{\mu\nu} = f^{-1}(\lambda)\left(\frac{1}{2} T^{(m)}_{\mu\nu} + \frac{1}{2} T^{(\lambda)}_{\mu\nu} + \nabla_\mu \nabla_\nu f - g_{\mu\nu} \Box f\right)}$$

modified Einstein Eq$^{\underline{n}}$ tend to arise from string theoretic derivations of G.R.

- Other alternative theories involve:
  - extra spatial dimensions (Kaluza Klein theory etc...)
  - higher-order terms in action
  - non-Christoffel connections (nontrivial torsion etc...)

- COSMOLOGICAL CONSTANT (introduced by Einstein to create static cosmology, then fell out of favor for decades, only to be revived around the time I was in graduate school in Physics $\approx$ 2001

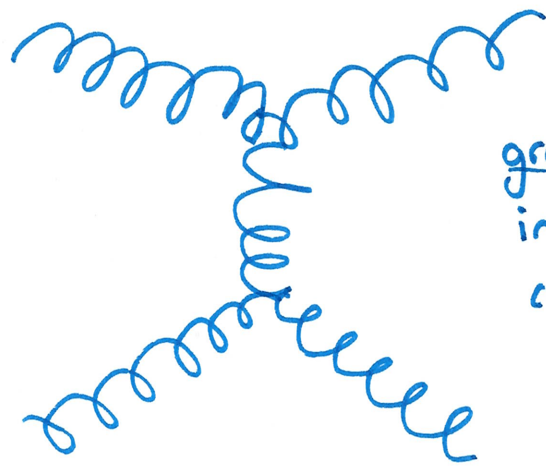
$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

As I understand it, the source of Λ is likely derived from some yet unknown speculative high energy physics. (But, $\Lambda \neq 0$ or $\Lambda = 0$ is perhaps within the scope of observational astronomy...)

PROPERTIES OF EINSTEIN'S EQUATION

(there is much more in Carroll, I'm picking the bigg. points)

- Gravitation is nonlinear:



gravitons interact with gravitons in contrast with E&M's force carrying particle the PHOTON

- Raychaudhuri's Equation (covered in Appendix F) reduces to $\frac{d\theta}{d\tau} = -4\pi G(\rho + P_x + P_y + P_z)$ which leads J.C. Baez to state: "The expansion of the volume of any set of particles initially at rest is proportional to (minus) the sum of the energy density and the three components of pressure"

- Einstein's Equations are nonlinear PDE's which are constrained by certain relations (1st order DEq^n) between Weyl tensor and the given stress-energy tensor... Moreover, various energy conditions (§4.6) are sometimes assumed to obtain physically reasonable solutions...
  (Carroll gives summary of some terminology §4.6)