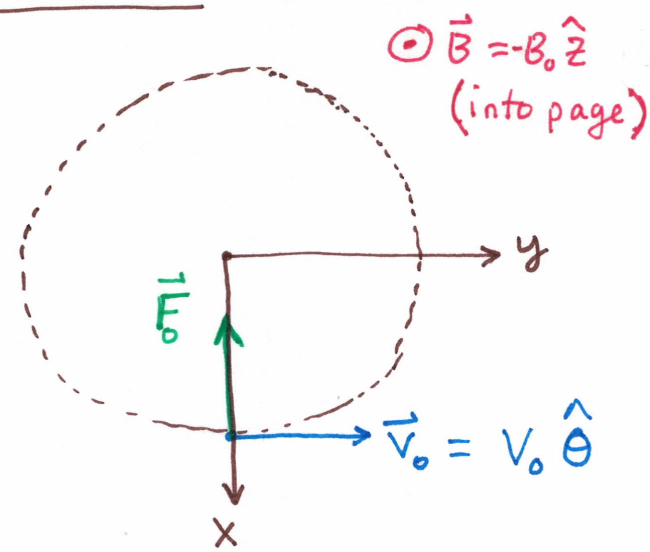
LECTURE 16 : MAGNETIC FIELDS

We begin our study of magnetostatics. We'll describe how steady currents give rise to magnetic fields and how magnetic fields place a force on moving charge. We begin with the Lorentz force

$$\vec{F} = Q(\vec{v} \times \vec{B})$$

If $\vec{E}$ and $\vec{B}$ are both involved then $\vec{F} = Q(\vec{E} + \vec{v} \times \vec{B})$
Griffiths mentions this law is due to Oliver Heaviside

$\boxed{E1}$ Given constant magnetic field $\vec{B} = -B_0 \hat{z}$
If we have a charge Q with initial velocity $\vec{v}_0 = v_0 \hat{y}$ then the resulting motion is circular. To keep things nice, let's assume the initial position is at $y_0 = z_0 = 0$ with $x_0 = \dfrac{m v_0}{B_0 Q}$



$$\vec{F} = Q(\vec{v} \times \vec{B}) = Q\left(v_0 \hat{\theta} \times (-B_0 \hat{z})\right) = -QB_0 v_0 \hat{r}$$

$$-\frac{m v_0^2}{R} \hat{r} = -QB_0 v_0 \hat{r} \Rightarrow \boxed{R = \frac{m v_0}{Q B_0}}$$

cyclotron formula

Remark : this formula assumes $v \ll c$, relativistically there is a correction.

PRODUCT RULES FOR $\vec{A} \times \vec{B}$ (from front cover of GRIFFITHS)

(6.) $\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$

(8.) $\nabla \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B} + \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A})$

Both of these flow from $\vec{A} \times \vec{B} = \sum_k' \epsilon_{ijk} A_i B_j \hat{x}_k$ and $\sum_k' \epsilon_{ijk} \epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$ and the product rule.

$$\begin{aligned}
\nabla \cdot (\vec{A} \times \vec{B}) &= \sum_k \partial_k (\vec{A} \times \vec{B})_k \\
&= \sum_k \partial_k \Big(\sum_{i,j}' \epsilon_{ijk} A_i B_j \Big) \\
&= \sum_{i,j,k} \epsilon_{ijk} \, \partial_k (A_i B_j) \\
&= \sum_{i,j,k} \epsilon_{ijk} (\partial_k A_i) B_j + \sum_{i,j,k} \epsilon_{ijk} A_i (\partial_k B_j) \\
&= \sum_j' \underbrace{\Big(\sum' \epsilon_{kij} \, \partial_k A_i \Big)}_{(\nabla \times \vec{A})_j} B_j - \sum_i' \underbrace{\Big(\sum_{k,j} \epsilon_{kji} \, \partial_k B_j \Big)}_{(\nabla \times \vec{B})_i} A_i \\
&= (\nabla \times \vec{A}) \cdot \vec{B} - (\nabla \times \vec{B}) \cdot \vec{A} \\
&= \underline{\vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})}. \quad (6.)
\end{aligned}$$

Continuing

$$\nabla \times (\vec{A} \times \vec{B}) = \sum_{i,j,k} \epsilon_{ijk} \, \partial_i (\vec{A} \times \vec{B})_j \, \hat{x}_k$$

$$= \sum_{i,j,k} \epsilon_{ijk} \, \partial_i \left(\sum_{l,m} \epsilon_{lmj} A_l B_m \right) \hat{x}_k$$

$$= \sum_{i,j,k,l,m} \epsilon_{ijk} \epsilon_{lmj} \left[(\partial_i A_l) B_m + A_l (\partial_i B_m) \right] \hat{x}_k$$

$$= \sum_{i,j,k,l,m} - \epsilon_{ikj} \epsilon_{lmj} \left[(\partial_i A_l) B_m + A_l (\partial_i B_m) \right] \hat{x}_k$$

$$= \sum_{i,k,l,m} - (\delta_{il} \delta_{km} - \delta_{im} \delta_{kl}) \left[(\partial_i A_l) B_m + A_l (\partial_i B_m) \right] \hat{x}_k$$

$$= \sum_{i,k,l,m} \Big(\delta_{im} \delta_{kl} (\partial_i A_l) B_m - \delta_{il} \delta_{km} (\partial_i A_l) B_m - \delta_{il} \delta_{km} A_l (\partial_i B_m) + \delta_{im} \delta_{kl} A_l (\partial_i B_m) \Big) \hat{x}_k$$

$$= \sum_{i,k} \Big(\underline{(\partial_i A_k) B_i} - \underline{(\partial_i A_i) B_k} - \underline{A_i \partial_i B_k} + \underline{A_k \partial_i B_i} \Big) \hat{x}_k$$

$$= \sum_{i,k} \Big(\underline{B_i \partial_i A_k} - \underline{A_i \partial_i B_k} + \underline{A_k \partial_i B_i} - \underline{(\partial_i A_i) B_k} \Big) \hat{x}_k$$

$$= \underline{(\vec{B} \cdot \nabla) \vec{A} - (\vec{A} \cdot \nabla) \vec{B} + \vec{A} (\nabla \cdot \vec{B}) - \vec{B} (\nabla \cdot \vec{A})} \quad (8.)$$

DERIVATION OF AMPERE'S LAW & NO MAGNETIC MONOPOLES

We begin by assuming the Biot-Savart law for a general volume current $\vec{J}$,

$$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'}$$

Where $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,' = (x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}$ and $d\tau' = dx'\,dy'\,dz'$ (primed coord. for source current, unprimed coord. at the field point $\vec{r} = (x,y,z)$)

$$\nabla\cdot\vec{B} = \frac{\mu_0}{4\pi}\int \nabla\cdot\left[\vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right]d\tau'$$

$$= \frac{\mu_0}{4\pi}\int\left[\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot\underbrace{(\nabla\times\vec{J})}_{0} - \vec{J}\cdot\underbrace{\nabla\times\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)}_{0}\right]d\tau'$$

$\vec{J} = \vec{J}(\vec{r}\,')$ Lemma from previous Lecture

$$\boxed{\nabla\cdot\vec{B} = 0 \quad \text{(no magnetic monopoles)}}$$

Continuing, the curl of $\vec{B}$ is specified by the calculation which follow. We <u>assume</u> Biot-Savart Law as our starting point and continue to use $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$ where <u>$\vec{J}(\vec{r}\,')$ has no $\vec{r}$-dependence</u> and $\nabla = \hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z$,

$$\nabla \times \left(\vec{J} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \underbrace{\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \cdot \nabla\right)\vec{J}}_{0} - \left(\vec{J} \cdot \nabla\right)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} + \vec{J}\left(\nabla \cdot \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) - \underbrace{\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\nabla \cdot \vec{J}}_{0}$$

Recall, $\nabla \cdot \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\vec{\mathscr{r}})$ then Biot-Savart,

$$\nabla \times \vec{B} = \frac{\mu_0}{4\pi}\int \nabla \times \left(\vec{J} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) d\tau'$$

$$= \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}\,') \cdot 4\pi\delta^3(\vec{r} - \vec{r}\,')\, d\tau' - \underbrace{\frac{\mu_0}{4\pi}\int \left(\vec{J} \cdot \nabla\right)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'}_{\text{can show this is zero see Lemma}}$$

$$= \mu_0 \vec{J}(\vec{r})$$

$$\boxed{\nabla \times \vec{B} = \mu_0 \vec{J} \qquad \text{Ampere's Law}}$$

Lemma: $-\int \left(\vec{J} \cdot \nabla\right)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau' = 0$

Observe $-\left(\vec{J} \cdot \nabla\right)\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \left(\vec{J} \cdot \nabla'\right)\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)$ since $\nabla' = \hat{x}'\frac{\partial}{\partial x'} + \hat{y}'\frac{\partial}{\partial y'} + \hat{z}'\frac{\partial}{\partial z'}$

and $\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\langle x - x', y - y', z - z' \rangle}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{3/2}}$ so switching from

$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$ for ∇' earns a minus.

Lemma proof continued

$$-(\vec{J}\cdot\nabla)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = (\vec{J}\cdot\nabla')\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)$$

$$= (\vec{J}\cdot\nabla')\left[\frac{1}{\mathscr{r}^3}(\vec{r}-\vec{r}\,')\right]$$

$$= (\vec{J}\cdot\nabla')\left\langle \frac{x-x'}{\mathscr{r}^3}, \frac{y-y'}{\mathscr{r}^3}, \frac{z-z'}{\mathscr{r}^3}\right\rangle$$

$$= \left\langle \underbrace{(\vec{J}\cdot\nabla')\left(\frac{x-x'}{\mathscr{r}^3}\right)}_{\nabla'\cdot\left[\frac{(x-x')}{\mathscr{r}^3}\vec{J}(\vec{r}\,')\right] - \left(\frac{x-x'}{\mathscr{r}^3}\right)\underbrace{\nabla'\cdot\vec{J}(\vec{r}\,')}_{\text{zero for steady currents}}}, (\vec{J}\cdot\nabla')\left(\frac{y-y'}{\mathscr{r}^3}\right), (\vec{J}\cdot\nabla')\left(\frac{z-z'}{\mathscr{r}^3}\right)\right\rangle$$

$$-(\vec{J}\cdot\nabla)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \left\langle \nabla'\cdot\left(\frac{x-x'}{\mathscr{r}^3}\vec{J}\right), \nabla'\cdot\left(\frac{y-y'}{\mathscr{r}^3}\vec{J}\right), \nabla'\cdot\left(\frac{z-z'}{\mathscr{r}^3}\vec{J}\right)\right\rangle$$

consequently,

$$-\int_{\mathcal{V}}(\vec{J}\cdot\nabla)\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)d\tau' = \left\langle \int_{\mathcal{V}} \nabla'\cdot\left(\frac{x-x'}{\mathscr{r}^3}\vec{J}(\vec{r}\,')\right)d\tau', \text{—}, \text{—}\right\rangle$$

$$= \left\langle \oint_{\partial\mathcal{V}}\left(\frac{x-x'}{\mathscr{r}^3}\right)\vec{J}\cdot d\vec{a}\,', \oint_{\partial\mathcal{V}}\left(\frac{y-y'}{\mathscr{r}^3}\right)\vec{J}\cdot d\vec{a}\,', \oint_{\partial\mathcal{V}}\left(\frac{z-z'}{\mathscr{r}^3}\right)\vec{J}\cdot d\vec{a}\,'\right\rangle$$

$$= 0$$

(take $\mathcal{V}$ large enough so that $\vec{J}=0$ on $\partial\mathcal{V}$, we assume $\vec{J}\neq 0$ only for some finite domain, although, we'll apply this for infinite line currents soon enough. See GRIFFITH'S § 5.3.1 for stand alone argument for such cases.)