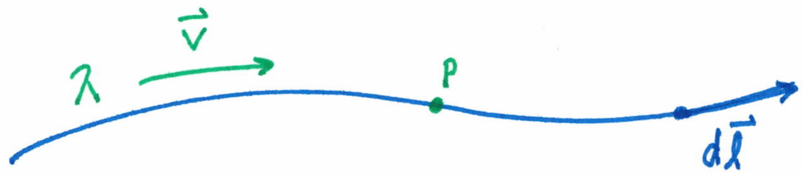
Lecture 17: Examples of Magnetic Fields

In the previous Lecture we saw Biot-Savart $\Rightarrow$ Ampere's Law. Now we'll take a step back to better understand the set-up of Biot-Savart and its application. Of course, Ampere's Law is prefered if symmetry permits. Begin with concept of current



current is charge per unit time passing a particular point P.

$$\vec{I} = \lambda \vec{V} \qquad \vec{V} = \frac{d\vec{l}}{dt}$$

$$\begin{aligned}
d\vec{F} &= (dQ)\vec{V} \times \vec{B} \\
&= (dQ)\frac{d\vec{l}}{dt} \times \vec{B} \\
&= \left(\frac{dQ}{dt}\right) d\vec{l} \times \vec{B} \\
&= \left(\frac{dQ}{dt}\right)(\hat{T}\, dl) \times \vec{B} \\
&= dl \left(\frac{dQ}{dt}\hat{T}\right) \times \vec{B} \\
&= dl\,(\vec{I} \times \vec{B})
\end{aligned}$$

$\frac{d\vec{l}}{dt} = v\hat{T}$

($\hat{T}$ points in direction of current)

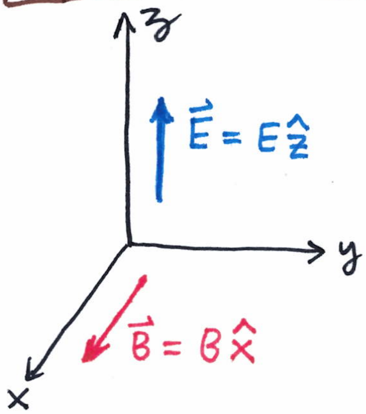
$$\boxed{\vec{F}_{mag} = \int (\vec{V} \times \vec{B})\, dQ = \int (\vec{I} \times \vec{B})\, dl = \int I (d\vec{l} \times \vec{B})}$$

Remark: magnetic forces do no work,

$$\vec{F}_{mag} \cdot \vec{V} = Q(\vec{V} \times \vec{B}) \cdot \vec{V} = 0.$$

E1 Consider case of constant $\vec{E}$ & $\vec{B}$ fields the motion is interesting

$$\vec{F} = Q(\vec{E} + \vec{v}\times\vec{B})$$



If the charge begins at the origin at $t=0$ at rest then we can argue $a_x = 0$ and the motion remains in the yz-plane.

$$\vec{V}\times\vec{B} = \langle V_2B_3 - V_3B_2,\ V_3B_1 - V_1B_3,\ V_1B_2 - V_2B_1\rangle = \langle 0, V_3B_1, -V_2B_1\rangle$$

$$\vec{V}\times\vec{B} = \langle 0, B\dot{z}, -B\dot{y}\rangle \quad \text{where } \vec{V} = \langle \dot{x}, \dot{y}, \dot{z}\rangle$$

$$\vec{F} = Q(E\hat{z} + B\dot{z}\hat{y} - B\dot{y}\hat{z}) = m(\ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z})$$

$$\vec{F} = m\vec{a} \longrightarrow \boxed{m\ddot{x} = 0,\quad m\ddot{y} = QB\dot{z},\quad m\ddot{z} = Q(E - B\dot{y})}$$

Introduce the cyclotron frequency $\omega = \frac{QB}{m}$ then we face

$$\boxed{\underbrace{\ddot{y} = \omega\dot{z}}_{①},\quad \underbrace{\ddot{z} = \omega\left(\tfrac{E}{B} - \dot{y}\right)}_{②}}$$

Let's try to work out the solution, we integrate ① to find

$$\underbrace{\dot{y} = \omega z + c_1}_{③} \Rightarrow \ddot{z} = \omega\left(\frac{E}{B} - \omega z - c_1\right)$$

$$\Rightarrow \ddot{z} + \omega^2 z = \frac{\omega E}{B} - c_1\omega$$

constant

$$\Rightarrow z = c_2\cos(\omega t) + c_3\sin(\omega t) + z_p$$

We need $\ddot{z}_p + \omega^2 z_p = \omega\left(\frac{E}{B} - c_1\right) \Rightarrow z_p = \frac{1}{\omega}\left(\frac{E}{B} - c_1\right)$.

$$z = c_2\cos(\omega t) + c_3\sin(\omega t) + \frac{1}{\omega}\left(\frac{E}{B} - c_1\right) \quad ④$$

$$\dot{y} = c_2\omega\cos\omega t + c_3\omega\sin\omega t + \frac{E}{B} \qquad (\text{from } ③)$$

$$y = c_2\sin\omega t - c_3\cos\omega t + \frac{Et}{B} + c_4 \quad ⑤$$

We found,

$$z = c_2 \cos \omega t + c_3 \sin \omega t + \frac{1}{\omega}\left(\frac{E}{B} - c_1\right)$$

$$y = c_2 \sin \omega t - c_3 \cos \omega t + \frac{Et}{B} + c_4$$

Now apply $y(0) = z(0) = 0$,

$$0 = c_2 + \frac{1}{\omega}\left(\frac{E}{B} - c_1\right) \quad \therefore \quad \underline{c_2 = \frac{1}{\omega}\left(c_1 - \frac{E}{B}\right)}.$$

$$0 = -c_3 + c_4 \quad \therefore \quad \underline{c_3 = c_4}.$$

Next, $\dot{y}(0) = \dot{z}(0) = 0$,

$$\dot{y}(0) = c_2 \omega + \frac{E}{B} = 0 \quad \therefore \quad c_2 = \frac{-E}{\omega B} = \frac{-E}{\omega B} + \frac{c_1}{\omega} \Rightarrow \underline{c_1 = 0}.$$

$$\dot{z}(0) = c_3 \omega = 0 \Rightarrow \underline{c_3 = 0}, \ \underline{c_4 = 0}. \qquad \text{and } \underline{c_2 = \frac{-E}{\omega B}}.$$

Therefore,

$$y(t) = \frac{-E}{\omega B}\sin(\omega t) + \frac{Et}{B}$$

$$z(t) = \frac{-E}{\omega B}\cos(\omega t) + \frac{E}{\omega B}$$

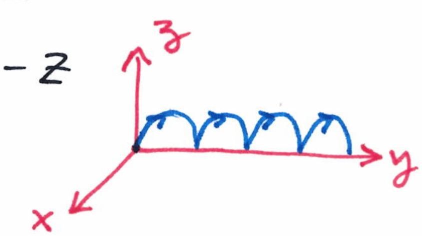
$$\boxed{\begin{aligned} y &= \frac{E}{B}\left(t - \frac{1}{\omega}\sin(\omega t)\right) = \frac{E}{\omega B}\left(\omega t - \sin(\omega t)\right) \\ z &= \frac{E}{\omega B}\left(1 - \cos(\omega t)\right) \end{aligned}}$$

Let $R = E/\omega B$ and use $\cos^2 \omega t + \sin^2 \omega t$ to simplify the above,
We find $y = R\omega t - R\sin(\omega t)$ and $z = R - R\cos(\omega t)$ hence

$$\sin(\omega t) = \frac{R\omega t - y}{R} \quad \text{and} \quad \cos(\omega t) = \frac{R - z}{R}$$

$$R\sin(\omega t) = R\omega t - y \quad \text{and} \quad R\cos(\omega t) = R - z$$

$$\boxed{(y - R\omega t)^2 + (z - R)^2 = R^2}$$



This is the equation of a circle of radius R with center $(0, R\omega t, R)$, we have a <u>cycloid</u>

CURRENT DISTRIBUTIONS NOTATION

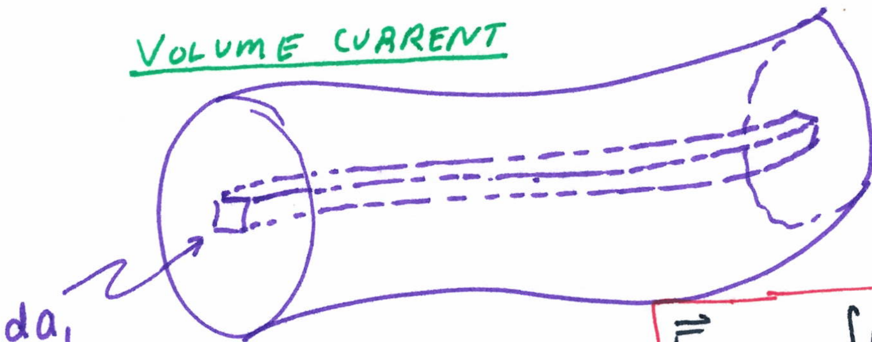
SURFACE CURRENT

$$\vec{K} \equiv \frac{d\vec{I}}{d\ell_\perp}$$

$$\vec{K} = \sigma \vec{v}$$

$$\boxed{\vec{F}_{mag} = \int (\vec{v} \times \vec{B})\sigma\, da = \int (\vec{K} \times \vec{B})\, da}$$

VOLUME CURRENT



$$\vec{J} \equiv \frac{d\vec{I}}{da_\perp}$$

$$\vec{J} = \rho \vec{v}$$

$$\boxed{\vec{F}_{mag} = \int (\vec{v} \times \vec{B})\rho\, d\tau = \int (\vec{J} \times \vec{B})\, d\tau}$$

LINE CURRENT

$$\vec{I} = \lambda \vec{v}$$

$$\boxed{\vec{F}_{mag} = \int (\vec{v} \times \vec{B})\lambda\, d\ell = \int (\vec{I} \times \vec{B})\, d\ell}$$

Remark: the boxed equation above indicate how to calculate the magnetic force on the given current subject a magnetic field $\vec{B}$ acting on the given current distribution.

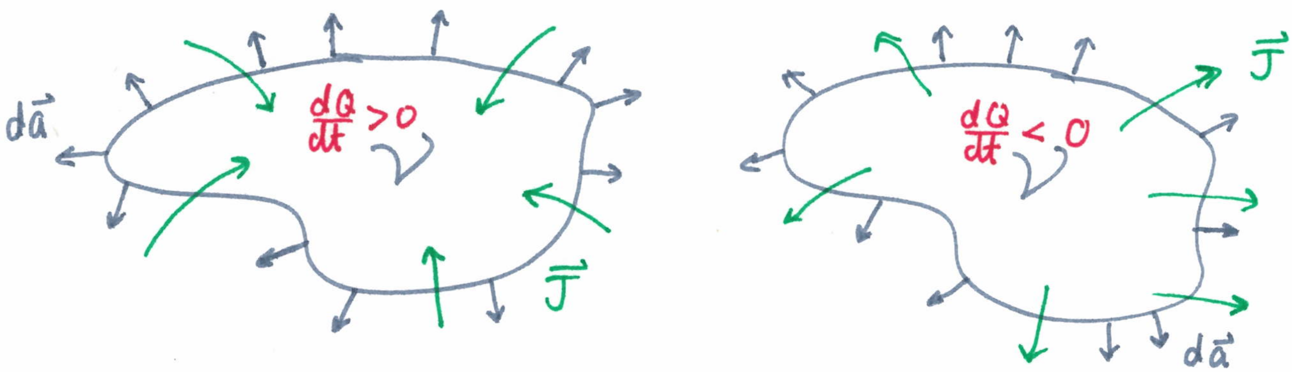
Conservation of Charge & The Continuity Equation

Consider a volume $\mathcal{V}$ with positively oriented boundary $\partial\mathcal{V}$. Notice the total charge Q within $\mathcal{V}$ is given by $Q = \int_{\mathcal{V}} \rho \, d\tau$ thus,

$$\frac{dQ}{dt} = \frac{d}{dt}\int_{\mathcal{V}} \rho \, d\tau = \int_{\mathcal{V}} \frac{\partial \rho}{\partial t} \, d\tau$$

If Q is changing then, since charge is conserved, there must be charge flowing through $\partial\mathcal{V}$, moreover whatever is lost in $\mathcal{V}$ must be gained outside $\mathcal{V}$

$$\frac{dQ}{dt} = -\int_{\partial\mathcal{V}} \vec{J} \cdot d\vec{a}$$



$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} d\tau = -\int_{\partial\mathcal{V}} \vec{J} \cdot d\vec{a} = -\int_{\mathcal{V}} (\nabla \cdot \vec{J}) \, d\tau$$

Since we can demand conservation of charge on <u>any</u> hypothetical volume $\mathcal{V}$ it follows that the charge density ρ and current density $\vec{J}$ must be connected by the <u>continuity equation</u>

$$\boxed{\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}}$$

<u>Remark</u>: this equation is the careful <u>defn</u> of charge conservation.

Biot-Savart Law and its Context

Magnetostatics is the study of problems with a constant magnetic field (in time, not space, to be clear). Such magnetic fields arise from steady currents

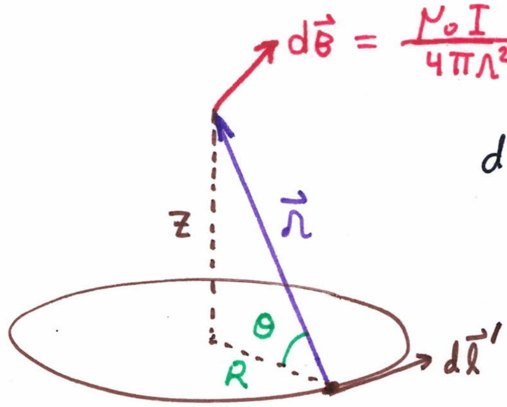
Defn/ A steady current is one for which charge is not accumulating at any point. Mathematically, $\frac{\partial \rho}{\partial t} = 0$ or equivalently $\nabla \cdot \vec{J} = 0$. And $\frac{\partial \vec{J}}{\partial t} = 0$

In the above context $\left(\frac{\partial \vec{J}}{\partial t} = 0 \text{ and } \nabla \cdot \vec{J} = 0 \right)$ the Biot-Savart Law describes the magnetic field due to a given current distribution $\vec{J}$, as usual $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$

$$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{I} \times \hat{\mathscr{r}}}{\mathscr{r}^2} \, dl' = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l}\,' \times \hat{\mathscr{r}}}{\mathscr{r}^2}}$$

Remark: $\mu_0 = 4\pi \times 10^{-7} \frac{N}{A^2}$ where $A = \frac{C}{S} = \frac{\text{Coulomb}}{\text{second}}$ is known as the permeability of space (we assume empty space)

E2 Suppose we have a steady current I going in a circular loop of radius R. Find $\vec{B}$ at point distance z above the center of the current loop.



$$d\vec{B} = \frac{\mu_0 I}{4\pi \mathscr{r}^2} d\vec{l}\,' \times \hat{\mathscr{r}} = \frac{\mu_0 I}{4\pi \mathscr{r}^3} d\vec{l}\,' \times \vec{\mathscr{r}}$$

use cylindrical coordinates

$$\begin{aligned} d\vec{l}\,' \times \vec{\mathscr{r}} &= (R\,d\theta\,\hat{\theta}) \times (-R\hat{s} + z\hat{z}) \\ &= -R^2 d\theta\, \hat{\theta} \times \hat{s} + zR\,d\theta\, \hat{\theta} \times \hat{z} \\ &= R^2 d\theta\, \hat{z} + zR\,d\theta\, \hat{r} \end{aligned}$$

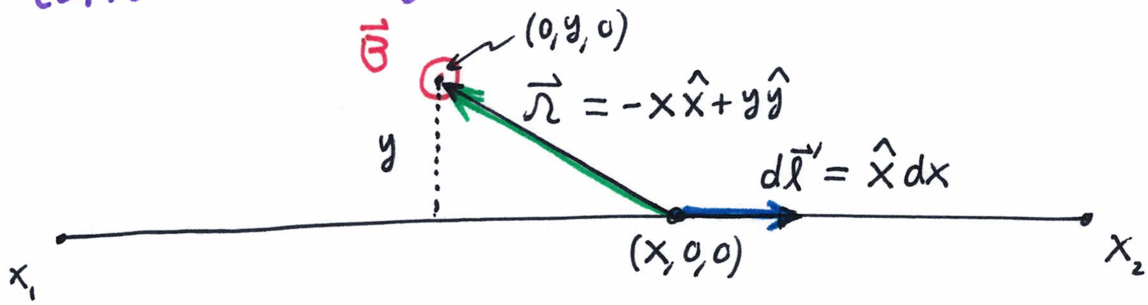
Notice $\mathscr{r}^2 = R^2 + z^2$ constant around loop,

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{1}{\mathscr{r}^3} (R^2 \hat{z} + zR\hat{r})\, d\theta = \frac{\mu_0 I R^2 (2\pi) \hat{z}}{4\pi \mathscr{r}^3}$$

$$\therefore \boxed{\vec{B} = \frac{\mu_0 I R^2 \hat{z}}{2(R^2 + z^2)^{3/2}}}$$

E3 consider a steady line current, find $\vec{B}$ field a distance s from the current.

For simplicity of calculation let's place the current I in the positive x-axis and let's place the field point at $(0, y, 0)$ where $y > 0$. We'll imagine the current running from $x = x_1$ to $x = x_2$.



$$d\vec{l}' \times \vec{\mathscr{r}} = (\hat{x}\,dx) \times (-x\hat{x} + y\hat{y}) = (y\,dx)\,\hat{z}$$

Consequently, $d\vec{B} = \dfrac{\mu_0 I}{4\pi \mathscr{r}^3}\, d\vec{l}' \times \vec{\mathscr{r}} = \left(\dfrac{\mu_0 I y\,dx}{4\pi \mathscr{r}^3}\right)\hat{z}$ and

$$\vec{B} = \left(\frac{\mu_0 I y \hat{z}}{4\pi}\right)\left(\int_{x_1}^{x_2} \frac{dx}{(x^2+y^2)^{3/2}}\right)$$

$x = y\tan\theta$

$dx = y\sec^2\theta\,d\theta$

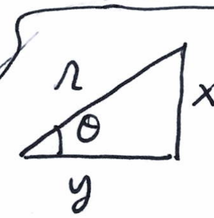
$x^2 + y^2 = y^2(\tan^2\theta + 1) = y^2\sec^2\theta$

$$= \left(\frac{\mu_0 I y \hat{z}}{4\pi}\right)\left(\int_{\theta_1}^{\theta_2} \frac{y\sec^2\theta\,d\theta}{(y^2\sec^2\theta)^{3/2}}\right)$$

$$= \left(\frac{\mu_0 I y \hat{z}}{4\pi}\right)\left(\int_{\theta_1}^{\theta_2} \frac{y\cos\theta}{y^3}\,d\theta\right)$$

$$= \frac{\mu_0 I \hat{z}}{4\pi y}\left(\sin\theta_2 - \sin\theta_1\right)$$

$$= \frac{\mu_0 I \hat{z}}{4\pi y}\left(\frac{x_2}{\sqrt{x_2^2+y^2}} - \frac{x_1}{\sqrt{x_1^2+y^2}}\right)$$



$\tan\theta = \dfrac{x}{y}$

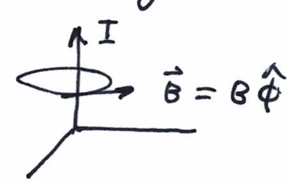
$\sin\theta = \dfrac{x}{\mathscr{r}}$

$\sin\theta_1 = \dfrac{x_1}{\sqrt{x_1^2+y^2}}$

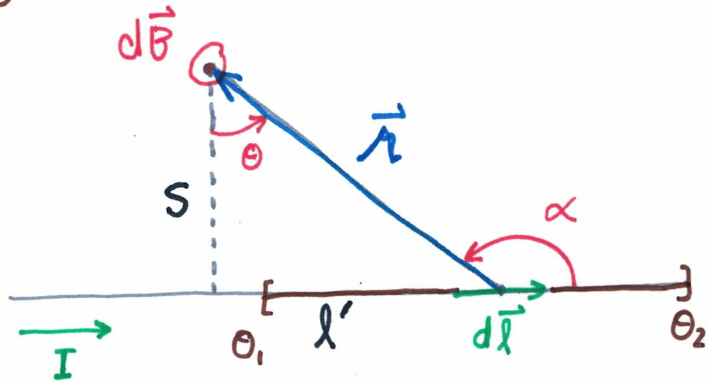
$\sin\theta_2 = \dfrac{x_2}{\sqrt{x_2^2+y^2}}$

Suppose $x_1 \longrightarrow -\infty$ and $x_2 \longrightarrow \infty$ then setting $y = s$ we obtain $\theta_1 = -\pi/2$ and $\theta_2 = \pi/2$ hence,

$$\boxed{\vec{B} = \frac{\mu_0 I \hat{\phi}}{2\pi s}}$$



$\boxed{E4}$ Straight line current, this time using GRIFFITH'S set-up



$d\vec{B}$ points out of page.

$$\| d\vec{\ell}' \times \hat{\mathscr{r}} \| = (\sin\alpha)\, d\ell' = \cos\theta\, d\ell'$$

$$\ell' = s\tan\theta$$

$$\underbrace{d\ell' = s \cdot \sec^2\theta\, d\theta}_{\text{since } s \text{ is fixed.}} = \frac{s\, d\theta}{\cos^2\theta}$$

Also, $s = \mathscr{r}\cos\theta$ thus

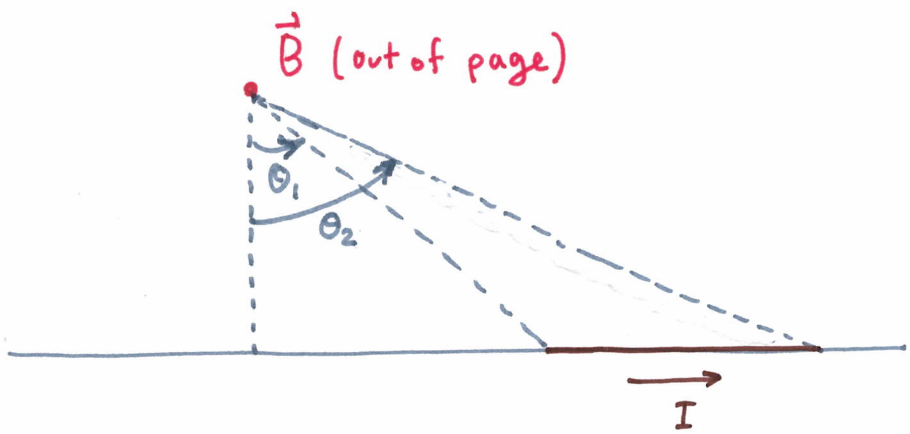
$$\frac{1}{\mathscr{r}^2} = \frac{\cos^2\theta}{s^2}$$

Consequently, the magnitude of the magnetic field,

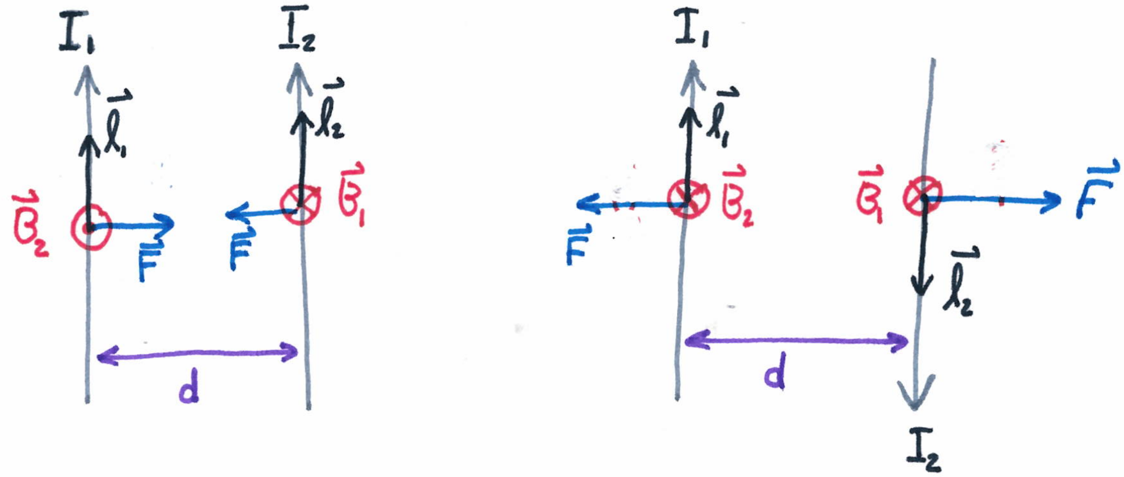
$$B = \frac{\mu_0 I}{4\pi} \int_{\theta_1}^{\theta_2} \left(\frac{\cos^2\theta}{s^2}\right)\left(\frac{s}{\cos^2\theta}\right) \cos\theta\, d\theta$$

$$= \frac{\mu_0 I}{4\pi s} \int_{\theta_1}^{\theta_2} \cos\theta\, d\theta$$

$$= \frac{\mu_0 I}{4\pi s}\left(\sin\theta_2 - \sin\theta_1\right)$$



E5 force between long linear currents



$\vec{F} = I\vec{l} \times \vec{B}$ (force on current I along $\vec{l}$ subject to field $\vec{B}$)

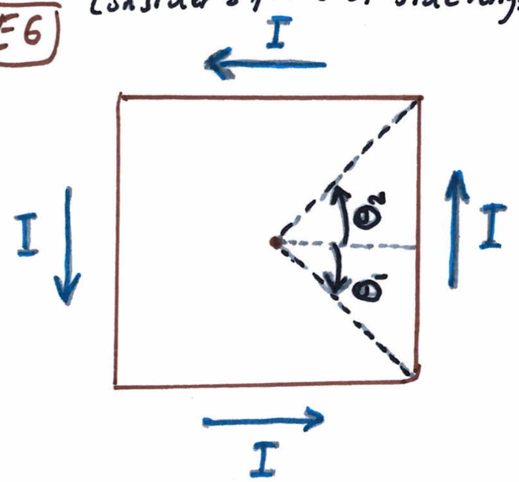
$$\|\vec{F}\| = \|I\vec{l} \times \vec{B}\| = IlB = I_1 l \frac{\mu_0 I_2}{2\pi d} \hookrightarrow \boxed{\frac{F}{l} = \frac{\mu_0 I_1 I_2}{2\pi d}}$$

force per length l for currents distance d apart.

(Remark: $\parallel$ currents attract
antiparallel currents repel.)

E6

Consider square of side length L, find $\vec{B}$ at center point.



$\theta_1 = -45°$ and $\theta_2 = 45°$

$$B_{\text{due to right segment}} = \frac{\mu_0 I}{4\pi\left(\frac{L}{2}\right)}\left(\sin 45° - \sin(-45°)\right)$$

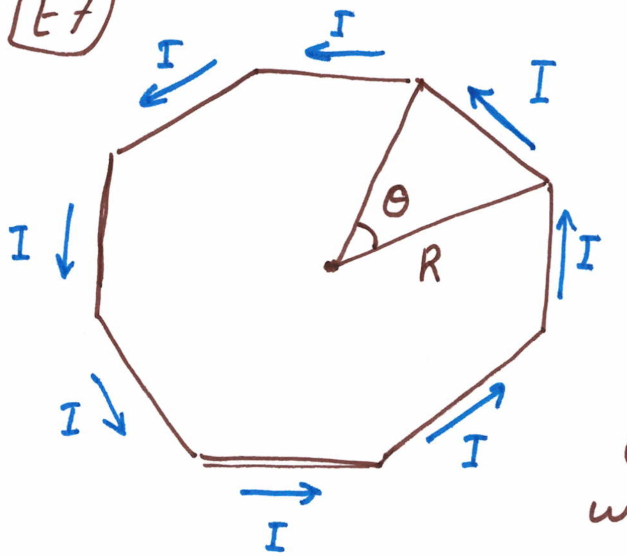
$$= \frac{\mu_0 I}{2\pi L}\left(\frac{2}{\sqrt{2}}\right)$$

$$= \frac{\mu_0 I}{\pi L\sqrt{2}}$$

Then by symmetry,

$$\boxed{B = 4\left(\frac{\mu_0 I}{\pi L\sqrt{2}}\right) = \frac{4\mu_0 I}{\pi L\sqrt{2}} \quad \text{(out of page)}}$$

$\boxed{E7}$



$$B_s = \frac{\mu_0 I}{4\pi R}\left[\sin\left(\frac{\theta}{2}\right) - \sin\left(-\frac{\theta}{2}\right)\right]$$

$$B_s = \frac{\mu_0 I}{2\pi R}\sin\left(\frac{\theta}{2}\right)$$

considering an n-sided polygon with "radius" R as pictured.

Notice $\theta = \frac{2\pi}{n}$ (for example, $\boxed{E6}$ is $n = 4$)

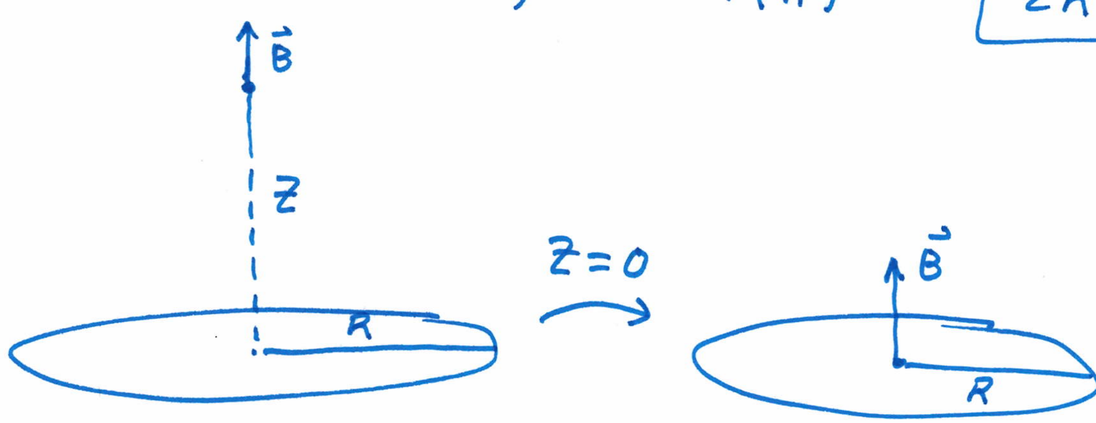
$$B = n B_s = \frac{\mu_0 I n}{2\pi R}\sin\left(\frac{2\pi}{2n}\right) \quad \text{out of page}$$

$$\boxed{B = \frac{\mu_0 I}{2R}\left(\frac{\sin(\pi/n)}{\pi/n}\right)}$$

Notice, as $n \longrightarrow \infty$ we find $\underline{\underline{B_\infty = \frac{\mu_0 I}{2R}}}$

Recall, from $\boxed{E2}$

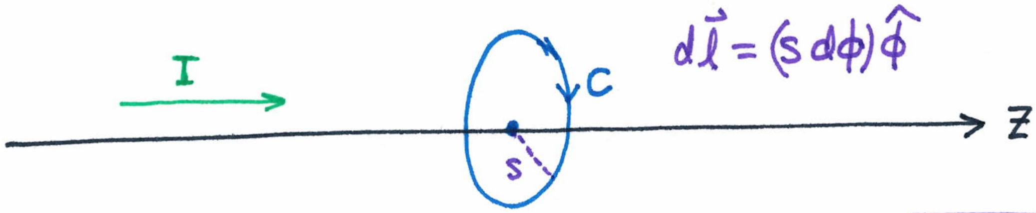
$$\|\vec{B}\| = \frac{\mu_0 I R^2}{2(R^2+z^2)^{3/2}} = \frac{\mu_0 I R^2}{2(R^2)^{3/2}} = \boxed{\frac{\mu_0 I}{2R}}$$



Examples via Ampere's Law: $\nabla\times\vec{B}=\mu_0\vec{J} \Rightarrow \boxed{\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}}$

E8 Once more consider the very long straight current I

$d\vec{l} = (s\,d\phi)\hat{\phi}$



$$\mu_0 I_{enc} = \underbrace{\oint_C \vec{B}\cdot d\vec{l} = 2\pi s B}_{} \Rightarrow \boxed{\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}}$$

since $\vec{B}$ must wrap around $\vec{I}$ following the right-hand-rule

$$\vec{B} = B\hat{\phi} \text{ and } \vec{B}\cdot d\vec{l} = (B\hat{\phi})\cdot(s\,d\phi\,\hat{\phi}) = Bs\,d\phi$$

$$\therefore \int_C \vec{B}\cdot d\vec{l} = Bs\int_0^{2\pi} d\phi = \underline{2\pi s B}.$$

E9 Constant current I spread over wire of radius R, again very long and we'll place it along z-axis

$$\vec{J} = \left(\frac{I}{\pi R^2}\right)\hat{z} \text{ for } 0 \leq s \leq R \text{ and } \vec{J} = 0 \text{ for } s > R.$$

Suppose C is circle of radius $\bar{s}$ where $0 \leq \bar{s} \leq R$ then

$$I_{enc} = \int_{D\,0}^{\bar{s}} \vec{J}\cdot d\vec{a} = \int_0^{\bar{s}}\int_0^{2\pi}\left(\frac{I}{\pi R^2}\right)(s\,d\phi\,ds) = \frac{I(\pi\bar{s}^2)}{\pi R^2}$$

There force,

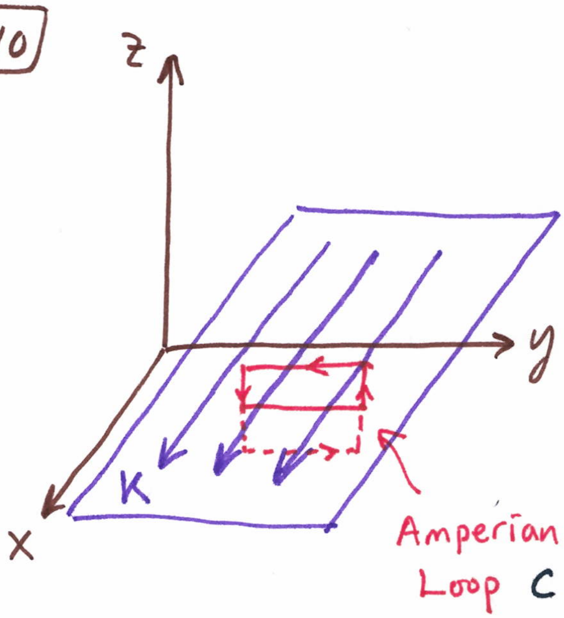
$$\mu_0 I_{enc} = \oint_C \vec{B}\cdot d\vec{l} \Rightarrow \frac{\mu_0 I \bar{s}^2}{R^2} = B(2\pi\bar{s})$$

It follows,

$$\Rightarrow \vec{B} = \frac{\mu_0 I \bar{s}}{2\pi R^2}\hat{\phi}$$

$$\boxed{\vec{B}(s) = \begin{cases} \dfrac{\mu_0 I s}{2\pi R^2}\hat{\phi} & : 0 \leq s \leq R \\ \dfrac{\mu_0 I}{2\pi s}\hat{\phi} & : s \geq R \end{cases}}$$

E10



$\vec{K} = K\hat{x}$

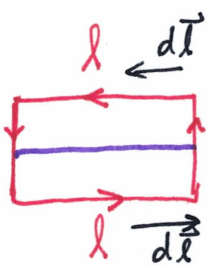
an infinite sheet current on the xy-plane.

- think about plane as union of ∞ly many linear currents in $\hat{x}$-direction

$z>0$ $\vec{B}$ points in $-\hat{y}$

$z<0$ $\vec{B}$ points in $\hat{y}$

By RHR and planar symmetry



$$\vec{B} = \begin{cases} -B\hat{y} & : z>0 \\ B\hat{y} & : z<0 \end{cases}$$ must match & be constant by planar symmetry.

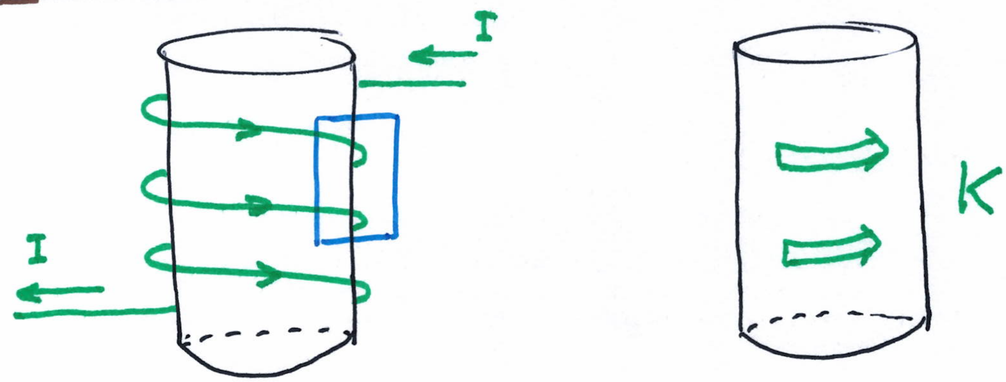
$$\oint \vec{B}\cdot d\vec{\ell} = 2B\ell = \mu_0 I_{enc} = \mu_0 K\ell$$

$$\therefore \quad B = \frac{\mu_0 K}{2}$$

$$\boxed{\vec{B} = \begin{cases} \left(\frac{\mu_0 K}{2}\right)\hat{y} & \text{for } z<0 \\ -\left(\frac{\mu_0 K}{2}\right)\hat{y} & \text{for } z>0 \end{cases}}$$

Remark: it might be instructive to formulate $\vec{B}$ in case $\vec{K}$ flows along plane with normal unit-field $\hat{n}$.

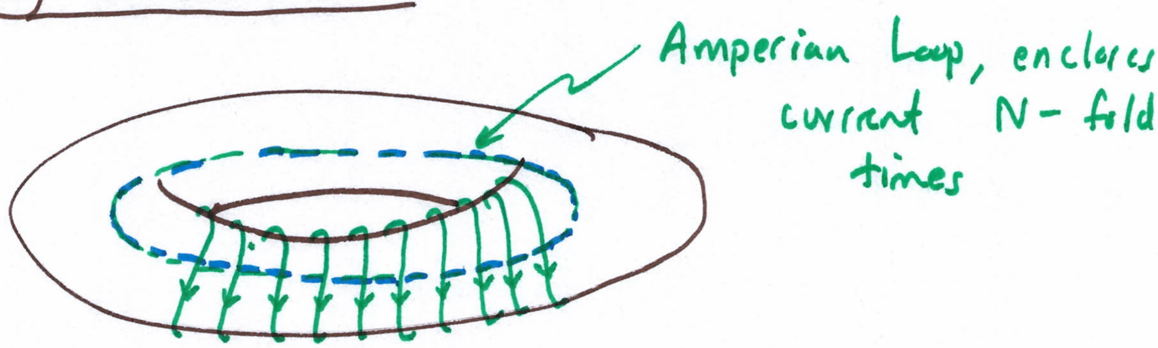
E11 solenoid



$$\oint \vec{B}\cdot d\vec{l} = BL = \mu_0 I_{enc} = \mu_0 n I L$$

$$\boxed{\vec{B} = \begin{cases} \mu_0 n I \hat{z} & : \text{inside solenoid} \\ 0 & : \text{outside solenoid} \end{cases}}$$

E12 toroidal solenoid



$$B(2\pi s) = \mu_0 I_{enc} = \mu_0 N I$$

$$\vec{B}(\vec{r}) = \begin{cases} \dfrac{\mu_0 N I}{2\pi s}\hat{\phi} & : \text{inside coil} \\ 0 & : \text{outside coil} \end{cases}$$