Lecture 17: The Schwarzschild Solution

Def^{n}/ The Schwarzschild metric in spherical coordinates $\{t, r, \theta, \phi\}$ is given by $ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$ where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ is the metric on unit-2 sphere.

This is the metric for a spherically symmetric gravitational field. It describes the Earth or Sun's gravitational field and it is the unique spherically symmetric metric.

Remark: for $M = 0$ we obtain
$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 = \underbrace{-dt^2 + dx^2 + dy^2 + dz^2}_{\text{Minkowski Metric}}$$

Sketch of Derivation

spherical symmetry $\longrightarrow$ $ds^2 = e^{-2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + e^{2\gamma(r)}r^2 d\Omega^2$

change to $\bar{r} = e^{\gamma(r)}r$ $\hookrightarrow$ $d\bar{r} = e^{\gamma}dr + e^{\gamma}r\,d\gamma = \left(1 + r\frac{d\gamma}{dr}\right)e^{\gamma}dr$

$\vdots$

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2$$

Remark: §5.2 on Birkhoff's Th^{m} derives the metric above from a fascinating symmetry argument based on foliations etc...

Given $ds^2 = -e^{2\alpha}dt^2 + e^{2\beta}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$ where α, β are fncts. of r alone.

$$\boxed{g_{tt} = -e^{2\alpha} \;\;\&\;\; g_{rr} = e^{2\beta} \;\;\&\;\; g_{\theta\theta} = r^2 \;\;\&\;\; g_{\phi\phi} = r^2\sin^2\theta}$$

Recall, $\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})$ then we can calculate the Christoffel symbols for the general spherically symmetric metric above,

$\Gamma^t_{tr} = \partial_r \alpha$	$\Gamma^r_{tt} = e^{2(\alpha-\beta)}\partial_r\alpha$	$\Gamma^r_{rr} = \partial_r\beta$
$\Gamma^\theta_{r\theta} = \frac{1}{r}$	$\Gamma^r_{\theta\theta} = -re^{-2\beta}$	$\Gamma^\phi_{r\phi} = \frac{1}{r}$
$\Gamma^r_{\phi\phi} = -re^{-2\beta}\sin^2\theta$	$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$	$\Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta}$

These are calculated by routine but lengthy calculus, for example,

$$\begin{aligned}
\Gamma^r_{rr} &= \frac{1}{2}g^{r\rho}(\partial_r g_{r\rho} + \partial_r g_{\rho r} - \partial_\rho g_{rr}) \\
&= \frac{1}{2}g^{rr}(\partial_r g_{rr} + \partial_r g_{rr} - \partial_r g_{rr}) \\
&= \frac{1}{2}\frac{1}{e^{2\beta}}\left(\frac{\partial}{\partial r}[e^{2\beta}]\right) \\
&= \frac{1}{2}\frac{1}{e^{2\beta}}\cdot e^{2\beta}\cdot 2\frac{\partial\beta}{\partial r} = \underline{\underline{\partial_r\beta}}.
\end{aligned}$$

Recall, $R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$ (Riemann Tensor) then calculating from the $\Gamma^{\sigma}_{\mu\nu}$ on pg. 2,

$$\boxed{\begin{aligned}
R^{t}{}_{rtr} &= (\partial_r\alpha)(\partial_r\beta) - \partial_r^2\alpha - (\partial_r\alpha)^2 \\
R^{t}{}_{\theta t\theta} &= -re^{-2\beta}\partial_r\alpha \\
R^{t}{}_{\phi t\phi} &= -re^{-2\beta}\sin^2\theta(\partial_r\alpha) \\
R^{r}{}_{\theta r\theta} &= re^{-2\beta}(\partial_r\beta) \\
R^{r}{}_{\phi r\phi} &= re^{-2\beta}\sin^2\theta(\partial_r\beta) \\
R^{\theta}{}_{\phi\theta\phi} &= (1-e^{-2\beta})\sin^2\theta
\end{aligned}}$$

$$\begin{aligned}
R_{\phi\phi} &= R^{\lambda}{}_{\phi\lambda\phi} \\
&= R^{t}{}_{\phi t\phi} + R^{r}{}_{\phi r\phi} + R^{\theta}{}_{\phi\theta\phi} \\
&= -re^{-2\beta}\sin^2\theta(\partial_r\alpha) \\
&\quad + re^{-2\beta}\sin^2\theta(\partial_r\beta) \\
&\quad + (1-e^{-2\beta})\sin^2\theta \\
&= \sin^2\theta\left[e^{-2\beta}(r(\partial_r\beta - \partial_r\alpha) - 1) + 1\right] \\
&= \sin^2\theta\, R_{\theta\theta}.
\end{aligned}$$

The Ricci Tensor is found from contracting the Riemann tensor according to $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$

$$\boxed{\begin{aligned}
R_{tt} &= e^{2(\alpha-\beta)}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - (\partial_r\alpha)(\partial_r\beta) + \frac{2}{r}\partial_r\alpha\right] \\
R_{rr} &= -\partial_r^2\alpha - (\partial_r\alpha)^2 + (\partial_r\alpha)(\partial_r\beta) + \frac{2}{r}\partial_r\beta \\
R_{\theta\theta} &= e^{-2\beta}\left[r(\partial_r\beta - \partial_r\alpha) - 1\right] + 1 \\
R_{\phi\phi} &= \sin^2\theta\, R_{\theta\theta}
\end{aligned}}$$

Finally, the Ricci Scalar, $R = g^{\mu\nu} R_{\mu\nu}$

$$R = -e^{-2\alpha} R_{tt} + e^{-2\beta} R_{rr} + \left(\frac{1}{r^2}\right) R_{\theta\theta} + \frac{1}{r^2 \sin^2\theta} R_{\phi\phi}$$

$$\boxed{R = -2e^{-2\beta}\left[\partial_r^2 \alpha + (\partial_r \alpha)^2 - (\partial_r \alpha)(\partial_r \beta) + \frac{2}{r}(\partial_r \alpha - \partial_r \beta) + \frac{1}{r^2}(1 - e^{2\beta})\right]}$$

The metric aims to describe gravity in the vacuum around a spherically symmetric mass/energy distribution. Einstein's field equation gives $R = 0$ in absence of mass/energy
Also, $R_{\mu\nu} = 0$ so we have $R_{tt} = 0$, $R_{rr} = 0$, $R_{\theta\theta} = 0$, $R_{\phi\phi} = 0$

$$0 = e^{2(\beta - \alpha)} R_{tt} + R_{rr} = \frac{2}{r}(\partial_r \alpha + \partial_r \beta) \Rightarrow \alpha + \beta = \text{constant}$$

$$0 = \frac{2}{r} \partial_r (\alpha + \beta) \qquad \underline{\alpha = -\beta + c}.$$

(α & β fncts of r alone)

Can rescale time coordinate, $ds^2 = -e^{2\alpha} dt^2 + e^{2\beta} dr^2 + r^2 d\Omega^2$
by $t \longmapsto e^{-c} t$ as to choose $c = 0$ and $\underline{\alpha = -\beta}$

$$R_{\theta\theta} = 0 \longrightarrow e^{2\alpha}(2r \partial_r \alpha + 1) = 1$$

$$\frac{\partial}{\partial r}(r e^{2\alpha}) = 1$$

$$r e^{2\alpha} = r + c \longrightarrow e^{2\alpha} = 1 + \frac{c}{r}$$

$$e^{2\alpha} = 1 - \frac{R_s}{r}$$ (following text for constant notation R_s)

Then we've brought the metric to the form

$$ds^2 = -\left(1 - \frac{R_s}{r}\right)dt^2 + \left(1 - \frac{R_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

All that remains is to fix the value of the constant R_s

Remark: we derived the formula $e^{2\alpha} = 1 - R_s/r$ from $R_{tt} + R_{rr} = 0$ and $R_{\theta\theta} = 0$. It turns out our soln also solves $R_{tt} = 0$ and $R_{rr} = 0$ for any value of the constant R_s

We derived that $g_{00} = -(1 + 2\Phi)$ in pg. 154 (weak field limit)

$$g_{tt} = -\left(1 - \frac{2GM}{r}\right) = -\left(1 - \frac{R_s}{r}\right) \Rightarrow \boxed{R_s = 2GM}$$

$$\boxed{ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2}$$

Defn $R_s = 2GM$ is the Schwarzschild Radius and ds^2 above is the Schwarzschild metric

SINGULARITIES:

The metric $ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$ is singular in the mathematical sense when $r = 2GM$ since the matrix of ds^2 is degenerate, the coeff. of dt^2 is zero and the coeff. of dr^2 gives division by zero. Is this physical? Is it a coordinate effect (for example ↘)

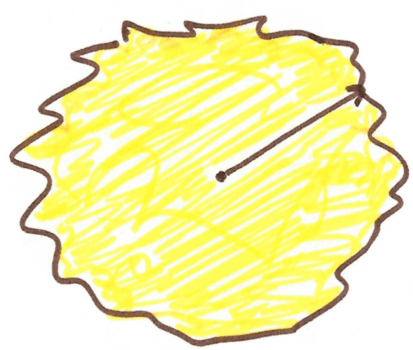
$\mathbb{R}^2$: $dx^2 + dy^2 = \underbrace{dr^2 + r^2 d\theta^2}_{\text{degenerate at } r=0}$

Is there a singularity at $r = 2GM$? Actually, no.

$$R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \frac{48 G^2 M^2}{r^6} \Leftarrow \text{singularity at } r = 0$$

$r = 2GM$ is the event horizon

SUN:



$R_{sun} = 10^6 GM_\odot$

$r = 2GM_\odot < 10^6 GM_\odot = R_{sun}$

($\nexists$ event horizon inside sun, Schwarzschild breaks down inside mass distribution)

Geodesics and Orbits in Schwarzschild

$\Gamma^r_{tt} = \frac{GM}{r^3}(r-2GM)$	$\Gamma^r_{rr} = \frac{-GM}{r(r-2GM)}$	$\Gamma^t_{tr} = \frac{GM}{r(r-2GM)}$
$\Gamma^\theta_{r\theta} = \frac{1}{r}$	$\Gamma^r_{\theta\theta} = -(r-2GM)$	$\Gamma^\phi_{r\phi} = \frac{1}{r}$
$\Gamma^r_{\phi\phi} = -(r-2GM)\sin^2\theta$	$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$	$\Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta}$

Gives geodesic equations

$$\frac{d^2t}{d\lambda^2} + \frac{2GM}{r(r-2GM)}\frac{dr}{d\lambda}\frac{dt}{d\lambda} = 0$$

$$\frac{d^2r}{d\lambda^2} + \frac{GM}{r^3}(r-2GM)\left(\frac{dt}{d\lambda}\right)^2 - \frac{GM}{r(r-2GM)}\left(\frac{dr}{d\lambda}\right)^2 - (r-2GM)\left[\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2\right] = 0$$

$$\frac{d^2\theta}{d\lambda^2} + \frac{2}{r}\frac{d\theta}{d\lambda}\frac{dr}{d\lambda} - \sin\theta\cos\theta\left(\frac{d\phi}{d\lambda}\right)^2 = 0$$

$$\frac{d^2\phi}{d\lambda^2} + \frac{2}{r}\frac{d\phi}{d\lambda}\frac{dr}{d\lambda} + 2\frac{\cos\theta}{\sin\theta}\frac{d\theta}{d\lambda}\frac{d\phi}{d\lambda} = 0$$

Carroll argues for the conservation of energy and angular momentum along geodesics (pg. 206 – 208)

$$E = -K_\mu\frac{dx^\mu}{d\lambda} = \left(1 - \frac{2GM}{r}\right)\frac{dt}{d\lambda} \quad \& \quad L = r^2\frac{d\phi}{d\lambda}$$

Note $\epsilon = -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$ is constant along geodesic,

$$-\left(1-\frac{2GM}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1-\frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2 = -\epsilon$$

From which we can derive,

$$-E^2 + \left(\frac{dr}{d\lambda}\right)^2 + \left(1-\frac{2GM}{r}\right)\left(\frac{L^2}{r^2}+\epsilon\right) = 0$$

$$\boxed{\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = \mathcal{E}} \quad \boxed{\underbrace{V(r) = \frac{1}{2}\epsilon - \epsilon\frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3}}} \qquad \mathcal{E} = \frac{1}{2}E^2$$

- effective potential
- similar result in Newtonian Gravity modulo the last term $\frac{GML^2}{r^3}$ ← new

Circular orbits at critical # for potential,

$$\frac{dV}{dr} = \frac{\epsilon GM}{r^2} - \frac{2L^2}{2r^3} + \frac{3GML^2}{r^4}$$

Setting $\frac{dV}{dr} = 0$ yields

$$\boxed{\epsilon GM r_c^2 - L^2 r_c + 3GML^2\gamma = 0}$$

$\gamma = 1$ GR

$\gamma = 0$ Newtonian Gravity

- For $\gamma = 0$ get $r_c = \frac{L^2}{\epsilon GM}$
- For $\gamma = 1$ the effective potentials are $\approx$ for large r, but when $r \to 0$ the novel $\frac{1}{r^3}$ term dominates
- For $\underbrace{\epsilon = 0}_{\text{massless}}$, $\gamma = 1$ get $\boxed{r_c = 3GM}$
  - $3GM < r < 6GM$ unstable circular orbits exist
  - $r > 6GM$ stable circular orbits exist
  - can dip below $r = 3GM$ and come back, but not below $r = 2GM$.

EXPERIMENTAL TESTS (§5.5 pg. 212 — 218 of Carroll)

- deflection of light
- precession of perihelia (the point of closest approach of a planet to the Sun)
- gravitational redshift

From $L = r^2 \frac{d\phi}{d\lambda}$ we find $\left(\frac{d\phi}{d\lambda}\right)^{-2} = \frac{r^4}{L^2}$ which multiplying on effective pot. eq^n yields

$$\boxed{\left(\frac{dr}{d\phi}\right)^2 + \frac{1}{L^2} r^4 - \frac{2GM}{L^2} r^3 + r^2 - 2GMr = \frac{2\mathcal{E}}{L^2} r^4}$$

Make a $x = \frac{L^2}{GMr}$ substitution to obtain,

$$\left(\frac{dx}{d\phi}\right)^2 + \frac{L^2}{G^2M^2} - 2x + x^2 - \frac{2G^2M^2}{L^2} x^3 = \frac{2\mathcal{E}L^2}{G^2M^2}$$

Now, differentiate w.r.t. ϕ, and divide by $2\frac{dx}{d\phi}$ to obtain

$$\boxed{\frac{d^2x}{d\phi^2} - 1 + x = \frac{3G^2M^2}{L^2} x^2}$$

← RHS term absent in Newtonian calculation

Expand $x = x_0 + x_1$ where x_0 solves Newtonian $\frac{d^2x_0}{d\phi} - 1 - x_0 = 0$

and x_1 solves $\frac{d^2x_1}{d\phi^2} - 1 + x_1 = \frac{3G^2M^2}{L^2} x_0^2$

From Newtonian Mechanics, we know $\boxed{x_0 = 1 + e\cos\phi \text{ where } e = \text{eccentricity}}$

$e^2 = 1 - b^2/a^2$ — a = semi-major axis, b = semi-minor axis.

After some calculation,

$$x = 1 + e\cos\phi + \frac{3G^2M^2e}{L^2}\phi\sin\phi$$

$$\Rightarrow \Delta\phi = \frac{6\pi G^2M^2}{L^2} \approx \frac{6\pi GM}{c^2(1-e^2)a}$$

→ $\Delta\phi_{\text{Mercury}} = 43.0''/\text{century}$.

(read pg. 216 for details)

GRAVITATIONAL REDSHIFT

Consider stationary observer in Schwarzschild, the 4-velocity U^μ has $U^i = 0$. Then $U_\mu U^\mu = -1$

$$g_{\mu\nu} U^\mu U^\nu = -1 \longrightarrow g_{00} U^0 U^0 = -1$$

$$\therefore U^0 = -\left(1 - \frac{2GM}{r}\right) U^0 U^0 = -1$$

$$U^0 = \frac{1}{\sqrt{1 - \frac{2GM}{r}}}$$

Such an observer measures the frequency ω of light along a null geodesic $x^\mu(\lambda)$

$$\omega = -g_{\mu\nu} U^\mu \frac{dx^\nu}{d\lambda}$$

since $U^i = 0$

$$= -g_{00} U^0 \frac{dt}{d\lambda}$$

$$= -\sqrt{1 - \frac{2GM}{r}}\, \frac{dt}{d\lambda}$$

recall $E = \left(1 - \frac{2GM}{r}\right) \frac{dt}{d\lambda}$

$$= \frac{E}{\sqrt{1 - 2GM/r}}$$

Since E is constant along geodesics $\Rightarrow$ ω changes with r

- Photon emitted at r_1 and observed at r_2 has

$$\frac{\omega_2}{\omega_1} = \sqrt{\frac{1 - 2GM/r_1}{1 - 2GM/r_2}} \approx 1 - \frac{GM}{r_1} + \frac{GM}{r_2} \quad \text{for } r \gg 2GM$$

frequency goes down as $r_2 > r_1$

$\Rightarrow$ redshift (photons falling into body are blueshifted)

- detected 1960 by Pound and Rebka over 72 ft Physics building at Harvard.

SPACETIME

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \qquad \{t, r, \theta, \phi\}$$

$$ds^2 = \left(1 - \frac{2GM}{r}\right)(-dt^2 + dr^{*2}) + r^2 d\Omega^2 \qquad \underbrace{r^* = r + 2GM \ln\left(\frac{r}{2GM} - 1\right)}_{\text{tortoise coordinate}}$$

$v = t + r^*$

$u = t - r^*$

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dv^2 + (dv\,dr + dr\,dv) + r^2 d\Omega^2$$

go through $r = 2GM$ with future directed path (p. 221)

Eddington – Finkelstein coordinates

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)du^2 - (du\,dr + dr\,du) + r^2 d\Omega^2$$

go through $r = 2GM$ with past directed path (p. 223)

$$ds^2 = -\frac{1}{2}\left(1 - \frac{2GM}{r}\right)(dv\,du + du\,dv) + r^2 d\Omega^2$$

where $\frac{1}{2}(v-u) = r + 2GM \ln\left(\frac{r}{2GM} - 1\right)$ implicitly defines $r = r(u,v)$

Next, $u' = -\left(\frac{r}{2GM} - 1\right)^{1/2} \exp\left(\frac{r-t}{4GM}\right)$

$v' = \left(\frac{r}{2GM} - 1\right)^{1/2} \exp\left(\frac{r+t}{4GM}\right)$

$$\underbrace{ds^2 = \frac{-16G^3M^3}{r} e^{\frac{-r}{2GM}} (dv'du' + du'dv') + r^2 d\Omega^2}_{\text{non singular nature of event horizon finally manifest in these coord.}}$$

KRUSKAL COORDINATES

$$T = \frac{1}{2}(v' + u') = \sqrt{\frac{r}{2GM} - 1}\; e^{\frac{r}{4GM}} \sinh\left(\frac{t}{4GM}\right)$$

$$R = \frac{1}{2}(v' - u') = \sqrt{\frac{r}{2GM} - 1}\; e^{\frac{r}{4GM}} \cosh\left(\frac{t}{4GM}\right)$$

$$ds^2 = \frac{32G^3M^3}{r} e^{\frac{-r}{2GM}} (-dT^2 + dR^2) + r^2 d\Omega^2$$

where $T^2 - R^2 = \left(1 - \frac{r}{2GM}\right) e^{\frac{r}{2GM}}$

implicitly defines $r = r(T,R)$.

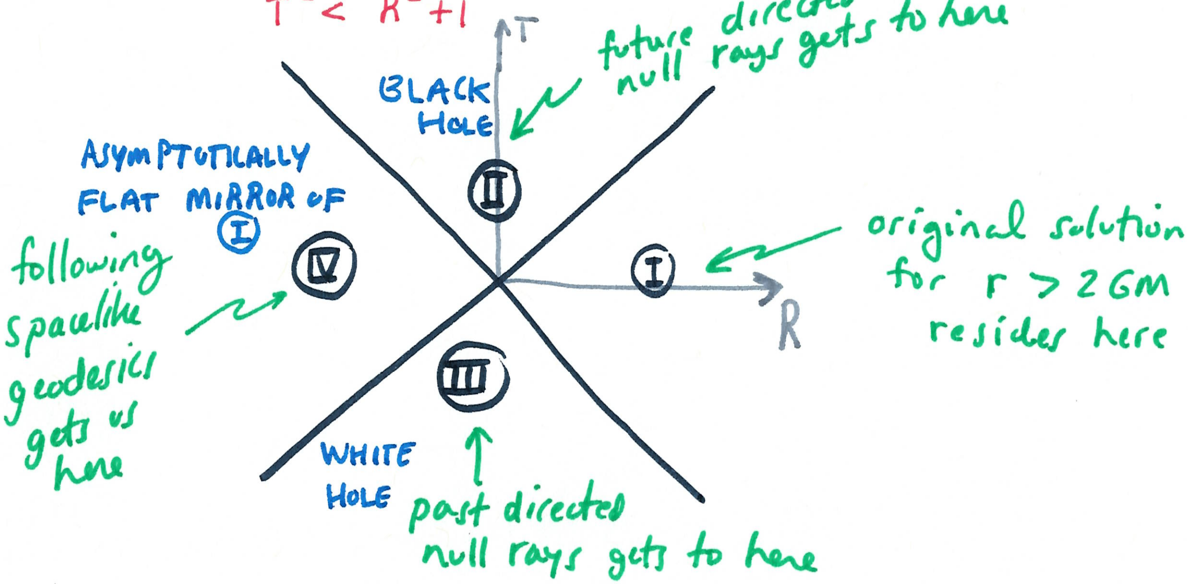
MAXIMALLY EXTENDED SCHWARZSCHILD SOLUTION

$$ds^2 = \frac{32G^3M^3}{r} e^{\frac{-r}{2GM}} \left(-dT^2 + dR^2\right) + r^2 d\Omega^2$$

Where $T^2 - R^2 = \left(1 - \frac{r}{2GM}\right) \exp\left(\frac{r}{2GM}\right)$ defines $r = r(T,R)$

Coordinates $\{T, R, \theta, \phi\}$ are KRUSKAL COORDINATES

- radial null curves follow $T = \pm R + \text{constant}$
- event horizon $r = 2GM$ given by $T = \pm R$
- constant r surfaces satisfy $T^2 - R^2 = \text{constant}$ (hyperbolae in T-R plane)
- constant t surfaces satisfy $\frac{T}{R} = \tanh\left(\frac{t}{4GM}\right)$ (lines through origin in TR-plane with slope $\tanh\left(\frac{t}{4GM}\right)$)
  $t \to \pm\infty$ this becomes $T = \pm R$
- $-\infty \leq R \leq \infty$
  $T^2 < R^2 + 1$



- can imagine wormhole to connect I and IV

- read p. 230
- read p. 235