## LECTURE 18: GRAVITATIONAL WAVES (CHAPTER 7 OF CARROLL)



Once again study a weak field limit of GR, this time not static

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad \text{where } |h_{\mu\nu}| << 1$$

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$$

(to $1^{st}$ order in $h$, note Carrol does entertain $2^{nd}$ formulae on p. 307–315 where he calculates energy loss due to gravitational radiation)

- Conceptually, think of flat space time with a field $h_{\mu\nu}$ which propagates

For instance, have local Lorentz transformations $h_{\mu'\nu'} = \Lambda_{\mu'}{}^{\mu} \Lambda_{\nu'}{}^{\nu} h_{\mu\nu}$

(for cosmology, to study gravitational waves you'd use a background metric $g^{(0)}_{\mu\nu}$)

- What are the equations of motion for the perturbation $h_{\mu\nu}$ to $\underline{1^{st} \text{ order}}$?

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}) = \frac{1}{2}\eta^{\rho\lambda}(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu})$$

Observe $\Gamma$ is $1^{st}$ order in $h$ thus consider only derivative terms for Riemann, the terms with $\Gamma$ contracted need not concern us,

$$R_{\mu\nu\rho\sigma} = \eta_{\mu\lambda}\partial_\rho \Gamma^{\lambda}_{\nu\sigma} - \eta_{\mu\lambda}\partial_\sigma \Gamma^{\lambda}_{\nu\rho}$$

$$= \eta_{\mu\lambda}\partial_\rho \left[\frac{1}{2}\eta^{\lambda\delta}(\partial_\nu h_{\sigma\delta} + \partial_\sigma h_{\delta\nu} - \partial_\delta h_{\nu\sigma})\right] - \left[\eta_{\mu\lambda}\partial_\sigma \Gamma^{\lambda}_{\nu\rho}\right]$$

$$= \frac{1}{2}\delta_{\mu\delta}\, \partial_\rho(\partial_\nu h_{\sigma\delta} + \partial_\sigma h_{\delta\nu} - \partial_\delta h_{\nu\sigma}) - \eta_{\mu\lambda}\partial_\sigma \Gamma^{\lambda}_{\nu\rho}$$

$$= \frac{1}{2}(\partial_\rho \partial_\nu h_{\sigma\mu} + \cancel{\partial_\rho\partial_\sigma h_{\mu\nu}} - \partial_\rho\partial_\mu h_{\nu\sigma}) - \frac{1}{2}\text{“}(\rho\sigma \to \sigma\rho)\text{”}$$

$$= \frac{1}{2}(\partial_\rho\partial_\nu h_{\sigma\mu} - \partial_\sigma\partial_\nu h_{\rho\mu} - \partial_\rho\partial_\mu h_{\nu\sigma} + \partial_\sigma\partial_\mu h_{\nu\rho})$$

cancel since $\partial_\rho\partial_\sigma = \partial_\sigma\partial_\rho$

Found $R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{\nu\rho} - \partial_\sigma\partial_\nu h_{\mu\rho} - \partial_\rho\partial_\mu h_{\nu\sigma}\right)$

Then $R_{\mu\nu} = \eta^{\alpha\beta} R_{\alpha\mu\beta\nu} = R^\sigma{}_{\mu\sigma\nu}$

Ricci Tensor →

$$\boxed{R_{\mu\nu} = \frac{1}{2}\left(\partial_\sigma\partial_\nu h^\sigma{}_\mu + \partial_\sigma\partial_\mu h^\sigma{}_\nu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}\right)}$$

$$\underbrace{\eta^{\alpha\beta}\partial_\alpha\partial_\beta}_{\Box} h_{\mu\nu}$$

Hence $R = \eta^{\mu\nu} R_{\mu\nu}$ yields the Ricci Scalar

$$\eta^{\mu\nu} R_{\mu\nu} = \frac{1}{2}\left(\partial_\sigma\partial_\nu h^{\sigma\nu} + \partial_\sigma\partial_\mu h^{\sigma\mu} - \Box h - \Box h\right)$$

$h = \eta^{\alpha\beta} h_{\alpha\beta} = h^\alpha{}_\alpha$

$$\therefore \quad \boxed{R = \partial_\mu\partial_\nu h^{\mu\nu} - \Box h}$$

Consequently, the Einstein Tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R$ gives

$$\boxed{G_{\mu\nu} = \frac{1}{2}\left(\partial_\sigma\partial_\nu h^\sigma{}_\mu + \partial_\sigma\partial_\mu h^\sigma{}_\nu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu}\partial_\rho\partial_\lambda h^{\rho\lambda} + \eta_{\mu\nu}\Box h\right)}$$

**Remark**: at this point Carroll enters a lengthy discussion on how to decompose $h_{\mu\nu}$ into several component objects which generally are highly interdependent... but ∃ several choices of gauge which we can fix to eliminate freedom in the equations as is convenient for various applications... I'll jump to the end where we learn on pg. 279 to define

$$h_{00} = -2\Phi \qquad \Psi = -\frac{1}{6}\delta^{ij} h_{ij}$$

$$h_{0i} = w_i \qquad S_{ij} = \frac{1}{2}\left(h_{ij} - \frac{1}{3}\delta^{kl} h_{kl}\delta_{ij}\right)$$

$$h_{ij} = 2S_{ij} - 2\Psi\delta_{ij}$$

where and $S_{ij}$ is traceless

Metric in terms of scalar $\Phi$, three vector $W_i$, traceless $S_{ij}$ and scalar $\Psi$

$$\boxed{ds^2 = -(1+2\Phi)dt^2 + W_i(dt\,dx^i + dx^i dt) + [(1-2\Psi)\delta_{ij} + 2S_{ij}]dx^i dx^j}$$

"Strain" contains the gravitational radiation as we shall see

The notation above is a decomposition of the metric perturbation $h_{\mu\nu}$. On pg. 280-283 Carrol examines the equations of motion for the fields $\Phi, W_i, S_{ij}, \Psi$ along geodesics as well as the form of the Einstein tensor $G_{\mu\nu}$ as it is expanded in terms of these fields e.g. $G_{00} = 2\nabla^2\Psi + \partial_k\partial_l S^{kl}$

transverse gauge $\quad \partial_i S^{ij} = 0$ and $\partial_i W^i = 0 \Rightarrow$

$$G_{00} = 2\nabla^2\Psi = 8\pi G T_{00}$$

Einstein's eq^ns

$$G_{0j} = -\frac{1}{2}\nabla^2 W_j + 2\partial_0\partial_j\Psi = 8\pi G T_{0j}$$

$$G_{ij} = (\delta_{ij}\nabla^2 - \partial_i\partial_j)(\Phi - \Psi) - \partial_0\partial_{(i}W_{j)} + 2\delta_{ij}\partial_0^2\Psi - \Box S_{ij} = 8\pi G T_{ij}$$

Synchronous gauge: $\Phi = 0$, $W^i = 0$

$$ds^2 = -dt^2 + (\delta_{ij} + h_{ij})dx^i dx^j$$

Lorentz/harmonic gauge:

$$\partial_\mu h^\mu{}_\nu - \frac{1}{2}\partial_\nu h = 0$$

Carrol then discusses $S^{ij} = S^{ij}_\perp + S^{ij}_s + S^{ij}_\parallel$ as it relates to $W^i = W^i_\perp + W^i_\parallel$ where

$\partial_i W^i_\perp = 0$ (divergenceless)

$\epsilon^{ijk}\partial_j W_{\parallel k} = 0$ (curl-free)

Gauge freedom in E&M

$$F = dA \qquad A' = A + d\lambda \rightarrow dA' = dA$$

## Newtonian Fields and Photon Trajectories

Work in rest frame of dust where $T_{\mu\nu} = \rho U_\mu U_\nu = \begin{pmatrix} \rho & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$

Einstein's Eq$^{=}$ in the transverse gauge, for static sources drop time-derivative terms and we obtain: (see pg. 3 and set a few terms to zero)

$$\nabla^2 \Psi = 4\pi G \rho$$
$$\nabla^2 w_j = 0$$
$$(\delta_{ij}\nabla^2 - \partial_i\partial_j)(\Phi - \Psi) - \nabla^2 S_{ij} = 0$$

We seek nonsingular solutions which behave well at $\infty$.

Hence, $w^i = 0$ and taking the trace of the $ij$-eq$^{=}$ get $2\nabla^2(\Phi - \Psi) = 0$ thus $\Phi = \Psi$. But, from our discussion in early chapters we know how $\Phi$ behaves in Newtonian limit... also true for $\Psi$ apparently? Not sure the point here on pg. 287, but, the following is clear,

$$\nabla^2 S_{ij} = 0 \implies \underline{S_{ij} = 0} \quad \text{for nice sol}^{=}$$

Anyway, seems like everything is zero, but $\Phi$ and $\Psi$ hence (again from 3)

$$\boxed{ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)(dx^2 + dy^2 + dz^2)}$$

$$h_{\mu\nu} = \begin{pmatrix} -2\Phi & & & \\ & -2\Phi & & \\ & & -2\Phi & \\ & & & -2\Phi \end{pmatrix} \qquad \nabla^2 \Phi = 4\pi G \rho$$

(in Chapter 4 we just found $h_{00} = -2\Phi$, now have $h_{ij}$ as well)

# Deflection via Gravity

$$x^\mu(\lambda) = \underbrace{x^{(0)\mu}(\lambda)}_{\text{solves geodesic eq}^{\text{n}}\text{ of background}} + x^{(1)\mu}(\lambda)$$

(see picture on p. 288) Figure 7.3 to define $\hat{\alpha}$

(this is path of photon or some other massless particle)

$$k^\mu \equiv \frac{dx^{(0)\mu}}{d\lambda} \qquad l^\mu \equiv \frac{dx^{(1)\mu}}{d\lambda}$$

$$g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0$$

$$\Rightarrow \eta_{\mu\nu} k^\mu k^\nu = 0$$

$$\Rightarrow (k^0)^2 = \vec{k} \cdot \vec{k} \equiv k^2$$

continuing, (see p. 289-291 for details)

$$\hat{\alpha} = 2 \int \vec{\nabla}_\perp \Phi \, ds$$

For point mass with background path along $x$,

$$\hat{\alpha} = 2GMb \int \frac{dx}{(b^2+x^2)^{3/2}} = \frac{4GM}{b}$$

← deflection of light

- Sir Arthur Eddington 1919, during total eclipse $\frac{4GM_\odot}{R_\odot c^2} \rightsquigarrow \hat{\alpha} = 1.75$ arcsecs
- Experiment maybe too imprecise to support claim
- gravitational lensing now an active area of astronomical research.
  (notice this gives a method to estimate mass which causes the deflection if everything else is pinned down ...)

# Gravitational Wave Solutions

"Turn off" the energy momentum tensor $T_{\mu\nu} = 0$ get $\nabla^2 \Psi = 0$ and $\nabla^2 w_j = 0$ and from the trace of $ij$ - eq: $\nabla^2 \Phi = 0$ thus only the strain tensor remains,

$$\boxed{\Box s_{ij} = 0}$$

In the transverse traceless gauge the perturbation metric $h^{TT}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & 2s_{ij} & \\ 0 & & & \end{pmatrix}$

$$\boxed{\Box h^{TT}_{\mu\nu} = 0 \quad \text{with} \quad \begin{array}{l} h^{TT}_{0\nu} = 0 \\ \eta^{\mu\nu} h^{TT}_{\mu\nu} = 0 \\ \partial_\mu h^{\mu\nu}_{TT} = 0 \end{array}}$$

complex-valued Thm

**Thm/** Plane wave solutions: $h^{TT}_{\mu\nu} = C_{\mu\nu} \exp(ik_\sigma x^\sigma)$ where $C_{\mu\nu}$ is a constant, symmetric, traceless (0,2) tensor | $C_{0\nu} = 0$, $\eta^{\mu\nu} C_{\mu\nu} = 0$

Proof: consider $\partial_\rho (\exp(ik_\sigma x^\sigma)) = \exp(ik_\sigma x^\sigma)\, \partial_\rho (ik_\sigma x^\sigma) = \exp(ik_\sigma x^\sigma) \cdot ik_\sigma \delta_{\rho\sigma}$ thus $\partial_\rho (\exp(ik_\sigma x^\sigma)) = ik_\rho \exp(ik_\sigma x^\sigma)$ thus,

$$\begin{aligned} \Box h^{TT}_{\mu\nu} &= \eta^{\alpha\beta} \partial_\alpha \partial_\beta [C_{\mu\nu} \exp(ik_\sigma x^\sigma)] \\ &= \underline{C_{\mu\nu}}\, \eta^{\alpha\beta} (ik_\alpha)(ik_\beta)\, \underline{\exp(ik_\sigma x^\sigma)} \\ &= -k^\rho k_\rho\, \underline{h^{TT}_{\mu\nu}} \end{aligned}$$

Thus $\Box h^{TT}_{\mu\nu} = 0$ requires, for nontrivial $h^{TT}_{\mu\nu}$, $\boxed{k^\rho k_\rho = 0}$

Solution is light-like.

- Write $k^\sigma = (\omega, k^1, k^2, k^3)$ then $k^\sigma k_\sigma = 0$ requires $\omega^2 = \delta_{ij} k^i k^j$ for our solution $h^{TT}_{\mu\nu} = C_{\mu\nu} \exp(i k_\sigma x^\sigma)$. Carroll remarks this solution is not unique & we can superpose only many such solutions

- To enforce transversality of the perturbation

$$0 = \partial_\mu h^{\mu\nu}_{TT} = i C^{\mu\nu} k_\mu e^{i k_\sigma x^\sigma} \longrightarrow \underline{k_\mu C^{\mu\nu} = 0}.$$

- Suppose wave travelling in the $x^3$-direction

$$k^\mu = (\omega, 0, 0, k^3) = (\omega, 0, 0, \omega)$$

Observe $k^\mu C_{\mu\nu} = 0$ and $C_{0\nu} = 0 \Rightarrow C_{3\nu} = 0$ hence only $C_{11}, C_{12}, C_{21}, C_{22}$ are nontrivial. Since $C_{\mu\nu}$ is both symmetric and traceless,

$$C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & C_{11} & C_{12} & 0 \\ 0 & C_{12} & -C_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

<u>Remark</u>: plane travelling in $x^3$-direction completely characterized by $C_{11}$, $C_{12}$ and $\omega$.

- On p. 296 Carroll describes how the plane wave moves nearby particles. The geodesic deviation equation yields (with appropriate approx. imposed)

$$\frac{\partial^2}{\partial t^2}(S^\mu) = \frac{1}{2} S^\sigma \frac{\partial^2}{\partial t^2} h^{TT\mu}{}_\sigma$$

: for wave travelling in $x^3$-direction only the $S^1$ and $S^2$ are disturbed by the wave (it is transverse)

Def$^{n}$/ $h_+ = C_{11}$ and $h_\times = C_{12}$ (read this as "cross" not x)
then we have,

$$C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

labels made to indicate motion caused by respective modes.

If $h_\times = 0$ then we should solve

$$\frac{\partial^2}{\partial t^2}(S^1) = \frac{1}{2} S^1 \frac{\partial^2}{\partial t^2}\left(h_+ e^{ik_\sigma x^\sigma}\right)$$

$$\frac{\partial^2}{\partial t^2}(S^2) = -\frac{1}{2} S^2 \frac{\partial^2}{\partial t^2}\left(h_+ e^{ik_\sigma x^\sigma}\right)$$

to find solutions of form,

$$\boxed{S^1 = \left(1 + \tfrac{1}{2} h_+ e^{ik_\sigma x^\sigma}\right) S^1(0) \quad \& \quad S^2 = \left(1 - \tfrac{1}{2} h_+ e^{ik_\sigma x^\sigma}\right) S_2(0)}$$

particles separated in $x^1$-direction have sep. in $x^1$-direction which oscillates in $x^1$-dir. Likewise $x^2$-separated particles likewise have oscillating $x^2$-separation ... gives oscillation in plus pattern (see figure 7.4 on pg. 297 for what $h_\times = 0$, $h_+ \neq 0$ plane wave does to ring of particles.)
Similarly, for $h_\times \neq 0$ and $h_+ = 0$ obtain

$$\boxed{S^1 = S^1(0) + \tfrac{1}{2} h_\times e^{ik_\sigma x^\sigma} S^2(0) \quad \& \quad S^2 = S^2(0) + \tfrac{1}{2} h_\times e^{ik_\sigma x^\sigma} S^1(0)}$$

the above solutions suggest separation oscillation in X-pattern, see pg. 298, FIG. 7.5

**Remark**: can see plane waves in terms of L & R polarized waves
$h_R = \frac{1}{\sqrt{2}}(h_+ + i h_\times)$ and $h_L = \frac{1}{\sqrt{2}}(h_+ - i h_\times)$. These are invariant under $180^\circ$ rotation $\Rightarrow$ the spin of graviton is 2

$$S = \frac{360^\circ}{\Theta}$$

Reminder: discuss p. 300 - 320 on product of waves & energy loss & detection.