Lecture 18: The Vector Potential

We begin with a comparison of electrostatics & magnetostatics

$$\underbrace{\boxed{\begin{aligned}\nabla\cdot\vec{E} &= \rho/\epsilon_0 \\ \nabla\times\vec{E} &= 0\end{aligned}}}_{\text{Electrostatics}} \qquad \underbrace{\boxed{\begin{aligned}\nabla\cdot\vec{B} &= 0 \\ \nabla\times\vec{B} &= \mu_0\vec{J}\end{aligned}}}_{\text{Magnetostatics}}$$

These are Maxwell's Equations in the special case of constant (in time) electric and magnetic fields. Griffith's explains that these paired with far field boundary conditions $E, B \to 0$ essentially define Electrostatics and Magnetostatics (except for our toy model infinite distribution examples like the infinite line current, or infinite line charge or infinite planar charge etc... yeah, so pretty much <u>all</u> the examples we usually test)

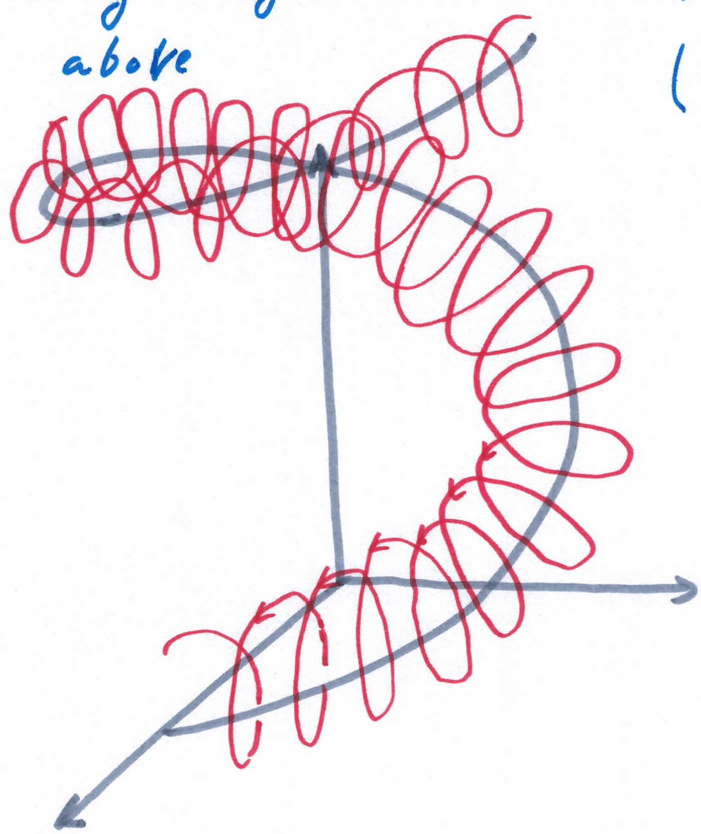
$$\underbrace{\vec{F} = Q(\vec{E} + \vec{V}\times\vec{B})}_{\text{Lorentz force law on } Q \text{ subject given } \vec{E} \,\&\, \vec{B}}$$

<u>Remark</u>: $\nabla\cdot\vec{B} = 0$ implies B-field lines cannot begin or terminate at some point $\left(\oint_{S=\partial V} \vec{B}\cdot d\vec{a} = \int_V (\nabla\cdot\vec{B})\,d\tau = 0 \right)$ Basically, however many field lines come into ∂V must also go out of ∂V. Somehow GRIFFITH's sees the existence of B-fields whose integral curves are not closed loops as evidence that B-field lines don't make sense. I am not convinced or worried by his concern.

However, the fact that $\exists$ current distributions for which the integral curves do not form closed loops is worthy of note. We often <u>incorrectly</u> teach $\vec{B}$ - field lines must come in closed loops. We should teach, B - field lines <u>cannot</u> <u>end</u>.

<u>CONJECTURE</u>: If current I follows along the helix $\vec{r}(t) = \langle R\cos t, R\sin t, ht \rangle$ then the resulting magnetic field exemplifies the claim above

(Biot-Savart here is rather difficult)



The Vector Potential

Defn/ Given a magnetic field $\vec{B}$ we say $\vec{A}$ is a <u>vector potential</u> for $\vec{B}$ if $\vec{B} = \nabla \times \vec{A}$

E1 Consider $\vec{B} = B_o \hat{z}$. To find a vector potential we need to solve:

$$\langle 0, 0, B_o \rangle = \langle \partial_y A_z - \partial_z A_y, \partial_z A_x - \partial_x A_z, \partial_x A_y - \partial_y A_x \rangle$$

Looks like we can get away with $A_z = A_x = 0$ hence reducing our task to

$$-\frac{\partial A_y}{\partial z} = 0 \quad \text{and} \quad \underbrace{\frac{\partial A_y}{\partial x} = B_o}_{\underline{A_y = B_o x}} \quad \text{solves } \frac{\partial A_y}{\partial z} = 0.$$

Hence,

$$\boxed{\vec{A}_1 = \langle 0, B_o x, 0 \rangle} \longrightarrow \vec{B} = \nabla \times \vec{A}_1 = B_o \hat{z} \ \checkmark$$

Let f be any function of x, y, z then

$$\vec{A}_2 = \langle 0, B_o x, 0 \rangle + \nabla f$$

is also a vector potential for $\vec{B}$!

$$\begin{aligned} \nabla \times \vec{A}_2 &= \nabla \times (\vec{A}_1 + \nabla f) \\ &= \nabla \times \vec{A}_1 + \nabla \times \nabla f \\ &= \vec{B} + \vec{0} \\ &= \vec{B}. \end{aligned}$$

The Point? there is vast freedom in our <u>choice</u> for $\vec{A}$.

Gauge Freedom

In both electrostatics and magnetostatics our choice of potential is not unique,

$$\vec{E} = -\nabla V = -\nabla(V + c)$$

$$\vec{B} = \nabla \times \vec{A} = \nabla \times (\vec{A} + \nabla f)$$

Note $V' = V + c$ and $\vec{A}' = \vec{A} + \nabla f$ illustrate gauge transformations of the scalar and vector potential.

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Poisson's Equations for Magnetism

Recall Ampere's Law $\nabla \times \vec{B} = \mu_0 \vec{J}$ and suppose $\vec{B} = \nabla \times \vec{A}$,

$$\nabla \times \vec{B} = \nabla \times (\nabla \times \vec{A})$$

$$= \underline{\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}}\ *$$

Defn/ If $\vec{B} = \nabla \times \vec{A}$ and $\nabla \cdot \vec{A} = 0$ then we say $\vec{A}$ is a vector potential for $\vec{B}$ in the Coulomb Gauge

(Griffith's Chap. 10 gives name)

Consequently, we find that:

Thm/ A vector potential for $\vec{B}$ generated by current density $\vec{J}$ (steady current) is given by

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$

provided we impose the Coulomb gauge $\nabla \cdot \vec{A} = 0$.

E2

Consider the magnetic field of long straight current I along the z-axis, we found from Ampere's Law that

$$\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

Let's try to derive a vector potential $\vec{A}$ via direct integration of $\vec{B} = \nabla \times \vec{A}$, use cylindricals

$$\frac{\mu_0 I}{2\pi s}\hat{\phi} = \left.\begin{aligned} &\left[\frac{1}{s}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\hat{s} \\ &+ \left[\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right]\hat{\phi} \\ &+ \frac{1}{s}\left[\frac{\partial}{\partial s}\left[s A_\phi\right] - \frac{\partial A_s}{\partial \phi}\right]\hat{z} \end{aligned}\right\} \nabla \times \vec{A}$$

We face the following PDEs,

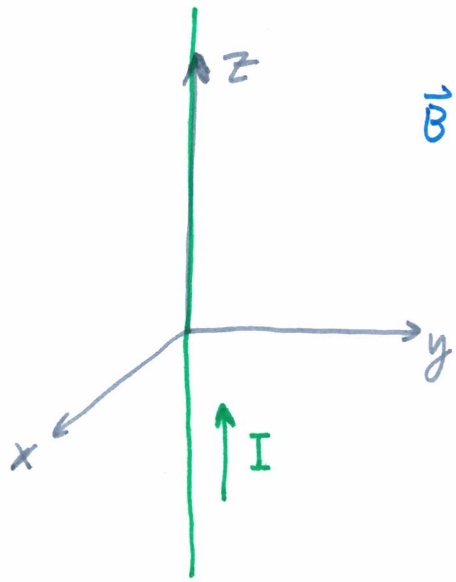
1. $\dfrac{1}{s}\dfrac{\partial A_z}{\partial \phi} - \dfrac{\partial A_\phi}{\partial z} = 0$
2. $\dfrac{1}{s}\left[\dfrac{\partial}{\partial s}\left[s A_\phi\right] - \dfrac{\partial A_s}{\partial \phi}\right] = 0$
3. $\dfrac{\partial A_s}{\partial z} - \dfrac{\partial A_z}{\partial s} = \dfrac{\mu_0 I}{2\pi s}$

Adhoc solution, $A_s = A_\phi = 0$ then 2 holds and we merely face $\dfrac{\partial A_z}{\partial \phi} = 0$ from 1 and $\dfrac{\partial A_z}{\partial s} = \dfrac{-\mu_0 I}{2\pi s}$

Hence $A_z = -\dfrac{\ln(s)\mu_0 I}{2\pi}$ and $\boxed{\vec{A} = \dfrac{-\mu_0 I}{2\pi}\ln(s)\,\hat{z}}$

We can check, $\nabla \times \vec{A} = \dfrac{\mu_0 I}{2\pi s}\hat{\phi}$.

E3 Suppose we find a $\overbrace{\text{scalar potential}}^{\Psi}$ for a magnetic field? Why is this troublesome? Let's consider a simple example,



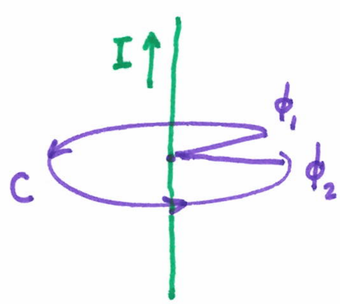
$$\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi} = -\nabla\Psi = -\hat{s}\frac{\partial\Psi}{\partial s} - \hat{z}\frac{\partial\Psi}{\partial z} - \frac{\hat{\phi}}{s}\frac{\partial\Psi}{\partial\phi}$$

We find solution with

$$\frac{\partial\Psi}{\partial s} = \frac{\partial\Psi}{\partial z} = 0$$

$$\frac{\mu_0 I}{2\pi s} = \frac{-1}{s}\frac{\partial\Psi}{\partial\phi}$$

$$\frac{d\Psi}{d\phi} = \frac{-\mu_0 I}{2\pi} \Rightarrow \boxed{\Psi = \frac{-\mu_0 I\phi}{2\pi}}$$



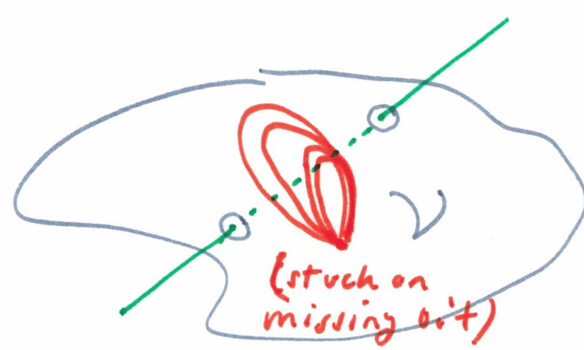
$$\int_C \vec{B}\cdot d\vec{l} = \frac{\mu_0 I}{2\pi}(\phi_2 - \phi_1)$$

If we go full-circle then $\phi_1 = \phi_2$ and $\oint_C \vec{B}\cdot d\vec{l} = 0$

Yet Ampere's Law gives $\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I$

We cannot hope to assign a scalar potential over any volume which is cut by a current. Note $\vec{B}$ not defined along the current hence the resulting domain $\mathcal{V}$ will not be simply connected

simply connected: can deform loops in $\mathcal{V}$ to point



if $\mathcal{V}$ is simply connected then apparently it is not domain for $\vec{B}$ with current piercing through.