Lecture 18: Tangent Spaces & Their Equations

pg. 206 – 212 in 2020 Lecture Notes.

Example 1: Let $x^3y^4 + z^3 = 0$ define a surface in $\mathbb{R}^3$. Find Eq$^{\underline{n}}$ of tangent plane at $(1,-1,-1)$.

$$F(x,y,z) = x^3y^4 + z^3$$

$$\nabla F = \langle 3x^2y^4,\ 4x^3y^3,\ 3z^2 \rangle$$

$$\nabla F(1,-1,-1) = \langle 3, -4, 3 \rangle \leftarrow \text{normal to tangent space}$$

$$\therefore \boxed{3(x-1) - 4(y+1) + 3(z+1) = 0}$$

$$x^2+y^2 = u^2 = z^2 \quad \text{cone}$$

$$\boxed{x^2+y^2 = z^2}$$

$$F(x,y,z) = x^2+y^2-z^2$$

Example 2 $\vec{r}(u,v) = \langle \overbrace{u\cos v}^{x}, \overbrace{u\sin v}^{y}, \overbrace{u}^{z} \rangle$ parametrizes surface for $0 \le u \le 1$ and $0 \le v \le 2\pi$. Find <u>normal</u> <u>vector</u> <u>field</u> and eq$^{\underline{n}}$ of tangent space at $\vec{r}(1, \pi/2)$.

$$\vec{N}(u,v) = \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \langle \cos v, \sin v, 1 \rangle \times \langle -u\sin v, u\cos v, 0 \rangle$$

$$= \det \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{bmatrix}$$

$$= \langle -u\cos v, -u\sin v, u\cos^2 v + u\sin^2 v \rangle$$

$u=1$
$v=\pi/2$

$$= \boxed{\langle -u\cos v, -u\sin v, u \rangle} \leftarrow \text{normal vector field.}$$

$$\vec{N}(1, \pi/2) = \langle 0, -1, 1 \rangle \text{ at } \vec{r}(1,\pi/2) = \langle 0, 1, 1 \rangle$$

$$0\cdot(x-0) - 1(y-1) + 1\cdot(z-1) = 0$$

$$\boxed{-y + z = 0}$$

$$F(x,y,z) = x^2 + y^2 - z^2$$

$$\nabla F = \langle 2x, 2y, -2z \rangle$$

$$\nabla F(0,1,1) = \underbrace{\langle 0, 2, -2 \rangle}_{\text{normal}} \;//\; \vec{N}(1, \pi/2) = \langle 0, -1, 1 \rangle$$

$$2(y-1) - 2(z-1) = 0 \quad // \quad -1(y-1) + 1\cdot(z-1) = 0$$

$$\boxed{2y - 2z = 0} \qquad \boxed{-y + z = 0}$$

Graph $(a, b, f(a,b))$

$$\vec{N} = \langle -f_x(a,b), -f_y(a,b), 1 \rangle$$

$$-f_x(a,b)(x-a) - f_y(a,b)(y-b) + 1\cdot(z - f(a,b)) = 0$$

$$\boxed{z = \underbrace{f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}_{L_f^{(a,b)}(x,y)}}$$

↑ linearization of $f(x,y)$