LECTURE 19: COSMOLOGICAL MODELS IN GR

(CHAPTER 8 IN CARROLL)

Copernican principle $\Rightarrow$ $\underbrace{\text{isotropy}}$ and $\underbrace{\text{homogeneity}}_{\mathbb{R}\times S^2}$

So, the models studied below are for math's sake, not physics.

Taken together these imply space is maximally symmetric (spacetime is not, past ≠ future ...)

$$\boxed{R_{\rho\sigma\mu\nu} = K\left(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}\right) \quad \text{where } K = \frac{R}{n(n-1)} \leftarrow \text{Ricci scalar}}$$

In this context R is constant over the manifold.

① $\underbrace{ds^2 = -dt^2 + dx^2 + dy^2 + dz^2}$ gives $K = 0$

Minkowski Space is maximally symmetric.

② de Sitter space: 4-dim'l slice of 5-D Minkowski space given by $-u^2 + x^2 + y^2 + z^2 + w^2 = \alpha^2$ with coordinates

$$\left.\begin{aligned} u &= \alpha \sinh(t/\alpha) \\ w &= \alpha \cosh(t/\alpha)\cos\chi \\ x &= \alpha \cosh(t/\alpha)\sin\chi\cos\theta \\ y &= \alpha \cosh(t/\alpha)\sin\chi\sin\theta\cos\phi \\ z &= \alpha \cosh(t/\alpha)\sin\chi\sin\theta\sin\phi \end{aligned}\right\} \quad ds^2 = -dt^2 + \alpha^2\cosh^2(t/\alpha)\underbrace{\left[d\chi^2 + \sin^2\chi\overbrace{\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)}^{d\Omega_2^2}\right]}_{d\Omega_3^2}$$

Carrol explains we can see this space is topologically $\underbrace{\mathbb{R}\times S^3}_{\text{Einstein's static universe}}$

3 **anti-de Sitter space**: begin with $\mathbb{R}^5$ with the type $(2,3)$-metric $ds_5^2 = -du^2 - dv^2 + dx^2 + dy^2 + dz^2$ and study the 4-dim'l space defined by: $-u^2 - v^2 + x^2 + y^2 + z^2 = -\alpha^2$

Define coordinates $\{t', \rho, \theta, \phi\}$ via:

$$\left.\begin{aligned} u &= \alpha \sin(t') \cosh\rho \\ v &= \alpha \cos(t') \cosh\rho \\ x &= \alpha \sinh(\rho) \cos\theta \\ y &= \alpha \sinh(\rho) \sin\theta \cos\phi \\ z &= \alpha \sinh(\rho) \sin\theta \sin\phi \end{aligned}\right\} \quad ds^2 = \alpha^2\left(-\cosh^2(\rho)\,dt'^2 + d\rho^2 + \sinh^2(\rho)\,d\Omega_2^2\right)$$

Carroll explains, in part by studying the space's "conformal diagram" that this space is topologically $\mathbb{R}^4$ with "∞ taking the form of a time like hypersurface"

<u>Remark</u>: most of pg. 326-327 is to motivate discussion of hot topic of ADS/CFT correspondence which links quantum gravity on ADS background to CFT on boundary...

<u>Relation to 1, 2 or 3 to gravitation?</u>

$$R_{\rho\sigma\mu\nu} = \kappa\left(g_{\rho\mu} g_{\sigma\nu} - g_{\rho\nu} g_{\sigma\mu}\right) \quad \text{where} \quad \kappa = \frac{R}{n(n-1)} = \frac{R}{4\cdot 3} = \frac{R}{12}$$

$$\begin{aligned} R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} &= \kappa\, g^{\lambda\alpha} R_{\alpha\mu\lambda\nu} \\ &= \kappa\, g^{\lambda\alpha}\left(g_{\alpha\lambda} g_{\mu\nu} - g_{\alpha\nu} g_{\mu\lambda}\right) \\ &= \kappa\left(g^{\lambda\alpha} g_{\lambda\alpha} g_{\mu\nu} - g_{\alpha\nu} g^{\lambda\alpha} g_{\mu\lambda}\right) \\ &= \kappa\left(4 g_{\mu\nu} - \delta_{\nu\lambda} g_{\mu\lambda}\right) = 3\kappa\, g_{\mu\nu} \end{aligned}$$

space where $R_{\mu\nu} \propto g_{\mu\nu}$ is called <u>Einstein space</u> (read paragraph above Eq. 8.18 on p. 328)

Found $R_{\mu\nu} = 3\boxtimes g_{\mu\nu}$ where $R = 12\boxtimes$

for maximally symmetric 4-manifolds (Minkowski, de Sitter, anti-de Sitter)

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$
$$= 3\boxtimes g_{\mu\nu} - \frac{1}{2}(12\boxtimes) g_{\mu\nu}$$
$$= -3\boxtimes g_{\mu\nu}$$

Then Einstein's field eq^{n}'s give $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ which imply,

$$\boxed{T_{\mu\nu} = \frac{-3\boxtimes}{8\pi G} g_{\mu\nu}}$$

→ vacuum energy or cosmological constant

$$\underset{\text{density}}{\rho} = -\underset{\text{pressure}}{P} = \frac{3\boxtimes}{8\pi G}$$

$\rho > 0 \Rightarrow$ de Sitter

$\rho < 0 \Rightarrow$ anti de Sitter

Carrol explains these spacetimes not consistent with observations for cosmology. However, these do serve as "ground states" for GR locally, they describe GR in absence of matter and gravitational radiation.

ROBERTSON - WALKER METRICS

Universe can be foliated into spacelike slices s.t. each slice is maximally symmetric so topologically spacetime is $\mathbb{R} \times \Sigma$ where Σ is max. sym. 3-manifold. Setting $d\sigma^2$ for the metric on Σ we suppose,

$$\boxed{ds^2 = -dt^2 + R^2(t)\, d\sigma^2}$$

$R(t)$ is the scale factor

$d\sigma^2 = \gamma_{ij}(u)\, du^i du^j$

$\underbrace{(u^1, u^2, u^3)}_{\text{so-called comoving coordinates}}$ coordinates on Σ

so-called <u>comoving</u> <u>coordinates</u>

The next page or so Carrol works on developing the necessary structure of $d\sigma^2$ given the max. sym. context. We'll follow along, (p. 329 - 330)

$$^{(3)}R_{ijkl} = k\left(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}\right) \quad \text{and} \quad k = \frac{^{(3)}R}{3(3-1)} = \frac{^{(3)}R}{6}$$

$$^{(3)}R_{jl} = g^{ik} R_{ijkl} = g^{ik}\left(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}\right) k$$
$$= \Big[\underbrace{(\gamma^{ik}\gamma_{ik})}_{3}\gamma_{jl} - \underbrace{\gamma^{ik}\gamma_{il}}_{\delta_{kl}}\gamma_{jk}\Big] k$$
$$= 2k\gamma_{jl}$$

Maximally Symmetric $\Rightarrow$ spherically symmetric

$$d\sigma^2 = e^{2\beta(\bar{r})} d\bar{r}^2 + \bar{r}^2 d\Omega^2$$

$$^{(3)}R_{11} = \frac{2}{\bar{r}}\partial_1\beta$$

$$^{(3)}R_{22} = e^{-2\beta}(\bar{r}\partial_1\beta - 1) + 1$$

$$^{(3)}R_{33} = \left[e^{-2\beta}(\bar{r}\partial_1\beta - 1) + 1\right]\sin^2\theta$$

$$2k\,\gamma_{11} = {}^{(3)}R_{11} = \frac{2}{\bar{r}}\partial_1\beta$$

$$2k\,\gamma_{22} = {}^{(3)}R_{22} = e^{-2\beta}(\bar{r}\,\partial_1\beta - 1) + 1$$

$$2k\,\gamma_{33} = {}^{(3)}R_{33} = [e^{-2\beta}(\bar{r}\,\partial_1\beta - 1) + 1]\sin^2\theta$$

Carroll claims these yield solution (I don't see it at the moment)

$$\beta = -\frac{1}{2}\ln(1 - k\bar{r}^2)$$

$$e^{2\beta} = e^{-\ln(1-k\bar{r}^2)} = \frac{1}{1-k\bar{r}^2}$$

$$\boxed{d\sigma^2 = \frac{d\bar{r}^2}{1-k\bar{r}^2} + \bar{r}^2 d\Omega^2}$$

$k = 0$ (flat)

$k = 1$ (positive curvature)

$k = -1$ (negative curvature)

Introduce new radial coordinate χ defined via $d\chi = \frac{d\bar{r}}{\sqrt{1-k\bar{r}^2}} \xrightarrow[\int]{} \bar{r} = S_k(\chi) = \begin{cases} \sin\chi & k=1 \\ \chi & k=0 \\ \sinh\chi & k=-1 \end{cases}$

$$\boxed{d\sigma^2 = d\chi^2 + S_k^2(\chi)\,d\Omega^2}$$

$k=0$: $d\sigma^2 = d\chi^2 + \chi^2 d\Omega^2 = dx^2 + dy^2 + dz^2$

$k=1$: $d\sigma^2 = d\chi^2 + \sin^2\chi\, d\Omega^2$ (metric of 3-sphere)

$k=-1$: $d\sigma^2 = d\chi^2 + \sinh^2\chi\, d\Omega^2$ (3-dim'l space of constant negative curvature)

Def^n/ metric on spacetime with maximally-symmetric hypersurfaces evolving in ~~time~~ size can be written,

$$ds^2 = -dt^2 + R^2(t)\left[\frac{d\bar{r}^2}{1-k\bar{r}^2} + \bar{r}^2 d\Omega^2\right]$$

ROBERTSON-WALKER METRIC

Can normalize variables to set $k = -1, 0, 1$

and introduce $a(t) = \frac{R(t)}{R_0}$ and $r = R_0 \bar{r}$ and $\kappa = \frac{k}{R_0^2}$ (takes any value)

$$\boxed{ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2\right]}$$

(Carroll's preferred formulation, consult my GR notes for other formulation)

From these we can calculate,

$$\Gamma^0_{11} = \frac{a\dot{a}}{1-\kappa r^2}$$

$$\Gamma^0_{22} = a\dot{a}r^2$$

$$\Gamma^1_{01} = \Gamma^2_{02}$$

$$\Gamma^1_{11} = \frac{\kappa r}{1-\kappa r^2}$$

$$\Gamma^3_{03} = \dot{a}/a$$

$$\Gamma^0_{33} = a\dot{a}r^2\sin^2\theta$$

$$\Gamma^1_{22} = -r(1-\kappa r^2)$$

$$\Gamma^1_{33} = -r(1-\kappa r^2)\sin^2\theta$$

$$\Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r}$$

$$\Gamma^2_{33} = -\sin\theta\cos\theta$$

$$\Gamma^3_{23} = \cot\theta$$

$$\underbrace{\begin{aligned} R_{00} &= -3\frac{\ddot{a}}{a} \\ R_{11} &= \frac{a\ddot{a} + 2\dot{a}^2 + 2\kappa}{1-\kappa r^2} \\ R_{22} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2\kappa) \\ R_{33} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2\kappa)\sin^2\theta \end{aligned}}_{\text{Ricci Tensor}}$$

$$\underbrace{R = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2}\right]}_{\text{Ricci Scalar}}$$

FRIEDMANN EQUATION

Investigate stress energy tensor's interplay with scale factor etc.
Model matter/energy as perfect fluid, in its rest frame $U^\mu = (1,0,0,0)$

$$T_{\mu\nu} = (\rho + P) U_\mu U_\nu + P g_{\mu\nu} \longrightarrow T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & & & \\ 0 & & g_{ij} P & \\ 0 & & & \end{pmatrix}$$

$$T^\mu{}_\nu = g^{\mu\alpha} T_{\alpha\nu} = g^{\mu 0} T_{0\nu} + g^{\mu j} T_{j\nu}$$

$$= \underbrace{-\delta_{\mu,0}}\, T_{0\nu} + \delta_{\mu,i} P \qquad (g^{\mu j} g_{ij} = \delta_{\mu,i})$$

$$ds^2 = -dt^2 + g_{ij}\, dx^i dx^j$$

$$T^\mu{}_\nu = \begin{pmatrix} -\rho & & & \\ & P & & \\ & & P & \\ & & & P \end{pmatrix} = \mathrm{diag}(-\rho, P, P, P)$$

$$T^\mu{}_\mu = -\rho + 3P = 0 \quad \text{for E\&M} *$$

$\hookrightarrow P_R = \frac{1}{3}\rho_R$
radiation ρ & P

Energy conservation, given by $0 = \nabla_\mu T^\mu{}_\nu$ so the $\nu = 0$ component is,

$$0 = \nabla_\mu T^\mu{}_0 = -\partial_0 \rho + \Gamma^\mu_{\mu\lambda} T^\lambda{}_0 - \Gamma^\lambda_{\mu 0} T^\mu{}_\lambda$$

$$\Rightarrow \boxed{0 = -\partial_0 \rho - 3\frac{\dot{a}}{a}(\rho + P)}$$

Hwk: calculate Γ^1_{01} and Γ^2_{02} and verify the boxed equation here.

Setting $P = w\rho$ the equation above yields:

$$\boxed{\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}} \xrightarrow[\text{constant}]{w} \quad \rho = c\, e^{-3(1+w)}$$

can integrate

Trace of electromagnetic stress-energy tensor $= 0$

$$T^{\mu\nu} = F^{\mu\lambda} F^{\nu}{}_{\lambda} - \frac{1}{4} g^{\mu\nu} F^{\lambda\sigma} F_{\lambda\sigma}$$

$$T^{\mu}{}_{\mu} = F^{\mu\lambda} F_{\mu\lambda} - \frac{1}{4} g^{\mu}{}_{\mu} F^{\lambda\sigma} F_{\lambda\sigma} \qquad g^{\mu}{}_{\mu} = g^{\mu\alpha} g_{\alpha\mu} = \delta_{\mu,\mu} = 4$$

$$T^{\mu}{}_{\mu} = F^{\mu\lambda} F_{\mu\lambda} - F^{\lambda\sigma} F_{\lambda\sigma} = 0$$

Carroll explains the scaling of matter and radiation via

$$\rho_M \propto a^{-3} \quad \text{and} \quad \rho_R \propto a^{-4}$$

Today, $\frac{\rho_m}{\rho_R} \sim 10^3$ earlier epoch the radiation density would be dominating:

$$\rho_\Lambda \propto a^0$$

(many cosmologies become vacuum-dominated if they don't eventually contract) e.g. de Sitter & anti-de Sitter

Recall Einstein's Eq^{n},

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)$$

Evaluating at $\mu = \nu = 0$ gives,

$$-3\frac{\ddot{a}}{a} = 8\pi G \left(T_{00} - \frac{1}{2} g_{00} T \right) = 8\pi G \left(\rho + \frac{1}{2}(-\rho + 3P) \right) = 4\pi G (\rho + 3P)$$

Likewise for $\mu\nu = ij$ we obtain,

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{K}{a^2} = 4\pi G (\rho - P)$$

$$\boxed{\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}}$$

Friedmann's Equation

&

$$\boxed{\frac{\ddot a}{a} = \frac{-4\pi G}{3}(\rho + 3P)}$$

Freedmann's 2nd Eq^n

Let's examine some well known descriptions of the physics here,

$\boxed{\text{Def}^n/\ H = \dot a / a \text{ is the Hubble parameter}}$ $\boxed{d_H = H_0^{-1} c}$ $\boxed{t_H = H_0^{-1}}$

Hubble distance — Hubble time

current measurements (2001) give $H = 70 \pm 10$ km/sec/Mpc

Mpc $= 3.09 \times 10^{24}$ cm

$\boxed{\text{Def}^n/\ q = -\frac{a\ddot a}{\dot a^2} \text{ the deceleration parameter}}$

$\boxed{\Omega = \frac{8\pi G}{3H^2}\rho = \frac{\rho}{\rho_{crit}} \text{ the density parameter}}$

Friedmann's Equation becomes $\Omega - 1 = \frac{k}{H^2 a^2}$

$\rho < \rho_{crit}$ $\Leftrightarrow$	$\Omega < 1$ $\Leftrightarrow$	$k < 0$ $\Leftrightarrow$	open
$\rho = \rho_{crit}$ $\Leftrightarrow$	$\Omega = 1$ $\Leftrightarrow$	$k = 0$ $\Leftrightarrow$	flat
$\rho > \rho_{crit}$ $\Leftrightarrow$	$\Omega > 1$ $\Leftrightarrow$	$k > 0$ $\Leftrightarrow$	closed

- Measurements put Ω very close to 1.
- Next $\approx$ 35 pages (340–374) concern the debate over the form of the early universe, including the add-on of inflation

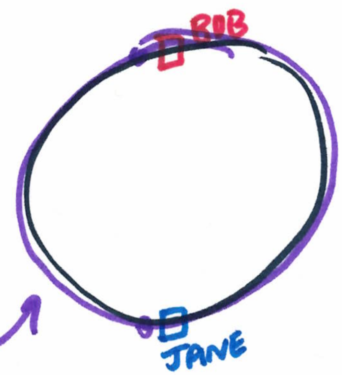
uniform expansion from initial singularity

$t=0$. ← BANG

$\underline{t = 3\text{min}}$

← soup of radiation, dense hot

↓ Cooling / Expanding



uniform CMB

- freeze out <u>matter</u>
- create CMB ← Pen. Wilson (1960 ?)

cosmic microwave background

~ 1978 or so, Alan Guth & ________ invented <u>inflationary</u> <u>cosmology</u>

∃ an <u>inflaton</u> <u>field</u>