Lecture 19: Monopole Field & Aharonov-Bohm

The distinction between electric and magnetic fields is perhaps more blurred than our common mathematical experience suggests. We saw the magnetic field and its vector potential are related via:

$$\boxed{\vec{B} = \nabla \times \vec{A}} \qquad \boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')}{\mathscr{r}}d\tau' = \frac{\mu_0}{4\pi}\int \frac{\vec{I}}{\mathscr{r}}dl'}$$

where $\vec{J}$ or $\vec{I}$ is the current distribution from which $\vec{B}$ arises. For decades physicists imagined magnetic monopoles as a means to analyze the source of B-fields. So far as we know no monopole has been observed. Let's examine how to construct the vector potential for a monopole,

E1 Suppose $\vec{B} = \dfrac{m\hat{r}\mu_0}{4\pi r^2}$ then $\nabla\cdot\vec{B} = \dfrac{m\mu_0}{4\pi}\nabla\cdot\left(\dfrac{\hat{r}}{r^2}\right) = m\delta^3(\vec{r})\mu_0$ would suggest $\vec{B}$ has m-units of magnetic charge. I'll omit $\dfrac{m\mu_0}{4\pi}$ and focus simply on $\hat{r}/r^2$,

$$\frac{\hat{r}}{r^2} = \nabla\times\vec{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{r}$$
$$+ \underbrace{\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}[rA_\phi]\right]\hat{\theta}}_{*}$$
$$+ \underbrace{\frac{1}{r}\left[\frac{\partial}{\partial r}[rA_\theta] - \frac{\partial A_r}{\partial\theta}\right]\hat{\phi}}_{**}$$

we need * and ** both zero.

Consider $A_r = A_\phi = 0$ then from $\hat{r}$ we see that

$$\frac{1}{r^2} = \frac{-1}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi} \Rightarrow \frac{\partial A_\theta}{\partial\phi} = \frac{-\sin\theta}{r} \Rightarrow A_\theta = \frac{-\phi\sin\theta}{r}$$

$$\boxed{\vec{A} = \left(\frac{-\phi\sin\theta}{r}\right)\hat{\theta}}$$

clearly $A_r = A_\phi = 0$ gives $* = 0$ and a short calculation shows $** = 0$ as well.

Remark: with $\vec{E} = \frac{Q\hat{r}}{4\pi\epsilon_0 r^2} = -\nabla\left(\frac{Q}{4\pi\epsilon_0 r}\right)$

we notice $V(\vec{r}) = \frac{Q}{4\pi\epsilon_0 r}$ is singular only

at the location of the charge. In contrast,

$\vec{A} = \left(\frac{-\phi \sin\theta}{r}\right)\hat{\theta}$ has unphysical singularities

connected with the angle-degeneracy or $\hat{\theta}$ not

being defined along z-axis. This is inevitable,

$$4\pi = \int_{S_R} \frac{\hat{r}}{r^2}\cdot d\vec{a} = \int_{S_R} (\nabla\times\vec{A})\cdot d\vec{a}$$

$$= \int_{B_R} \nabla\cdot(\nabla\times\vec{A})\,d\tau$$

$$= 0$$

not legit application because $\vec{A}$ has singularities within B_R

Remark: the above, or probably more properly what follows below is known as the DIRAC STRING

$$\vec{A} = \frac{g}{r(r-z)}\langle y, -x, 0\rangle$$

We see $r = z$ gives singularity for $\vec{A}$ along the positive z-axis. However, $\nabla\times\vec{A} = g\frac{\hat{r}}{r^2} = \vec{B}$ has only $r = 0$ as the physical singularity of $\vec{B}$.

(see §1.9 on Magnetic Monopoles in 2nd Ed. of GEOMETRY, TOPOLOGY AND PHYSICS by Mikio Nakahara, an excellent text for further reading on extensions of Electrodynamics to gauge theory etc.)

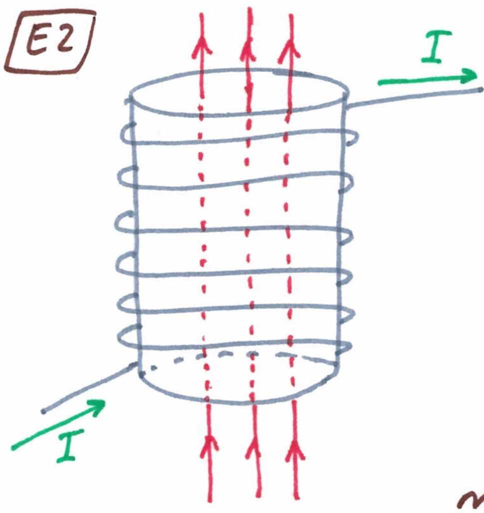
Aharonov - Bohm Effect

From Quantum Mechanics, a charge q moving along a path P where $\vec{B} = 0$ but $\vec{A} \neq 0$ has a <u>phase</u> <u>shift</u>

$$\boxed{\varphi = \frac{q}{\hbar} \int_P \vec{A} \cdot d\vec{l}}$$

This means $\vec{A}$ is measureable, let's see how,

E2



We imagine the solenoid is very long compared to its radius R then by the usual arguments

$$\vec{B} = \begin{cases} \mu_0 n I \, \hat{z} & 0 \leq s < R \\ 0 & s > R \end{cases}$$

from the homework, for a <u>constant</u> magnetic field we can derive the vector potential from:

$$\vec{A}(\vec{r}) = \frac{-1}{2}\left(\vec{r} \times \vec{B}\right) = \frac{-1}{2}\langle x, y, z\rangle \times \langle 0, 0, B\rangle$$

$$= \langle -yB/2,\ xB/2,\ 0\rangle$$

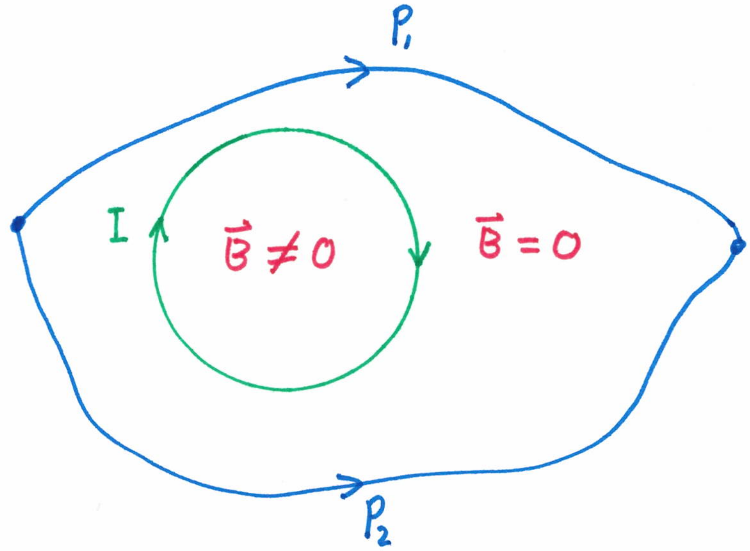
$$= \underline{\frac{sB}{2}\hat{\phi} \quad \text{for} \quad 0 \leq s \leq R}$$

Then by continuity, (which we should derive generally)

$$\underline{\vec{A}(\vec{r}) = \left(\frac{RB}{2}\right)\hat{\phi} \quad \text{for } s \geq R}$$

continued

E2 continued, let's examine how the magnetic field $B = \mu_0 n I$ within solenoid influences phase developed via $\varphi = \frac{q}{\hbar}\int \vec{A}\cdot d\vec{l}$



$$\vec{B} = \begin{cases} \mu_0 n I \hat{z} & 0 \leq s < R \\ 0 & s > R \end{cases}$$

$$\vec{A} = \begin{cases} \dfrac{\mu_0 n I s}{2} \hat{\phi} & : 0 \leq s \leq R \\ \dfrac{\mu_0 n I R}{2} \hat{\phi} & : s > R \end{cases}$$

Let S be region bounded by P_1 and P_2 ; $\partial S = P_2 \cup (-P_1)$ then by Stokes' Th^{m}

$$\int_{\partial S} \vec{A}\cdot d\vec{l} = \int_S (\nabla \times \vec{A})\cdot d\vec{a}$$

$$= \int_S \vec{B}\cdot d\vec{a}$$

$$= (\pi R^2)\mu_0 n I = \int_{P_2} \vec{A}\cdot d\vec{l} - \int_{P_1} \vec{A}\cdot d\vec{l}$$

$$\therefore \underbrace{\boxed{\varphi_2 - \varphi_1 = \pi R^2 \mu_0 n I}}$$

difference in phase of moving charge q happens along paths where $\vec{B} = 0$ for whole path.