Lecture 1: Vectors & Calculus

We review vector formalism and I share some notational devices which save much writing.

Def^n/ $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ <u>Kronecker</u> <u>Delta</u>

ϵ_{ijk} is defined to be completely antisymmetric in any exchange of index pair and $\epsilon_{123} = 1$

$$\epsilon_{ijk} = -\epsilon_{jik} \quad \text{and} \quad \epsilon_{ijk} = -\epsilon_{ikj} \quad \text{and} \quad \epsilon_{ijk} = -\epsilon_{kji}$$

Thus explicitly,

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$$

$$\epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1$$

Then we can formulate orthonormality

Def^n/ $\{V_1, .., V_n\}$ is orthonormal if $V_i \cdot V_j = \delta_{ij}$.

or, define the standard Cartesian unit-vectors

$$(\hat{x}_i)_j = \delta_{ij} \quad \rightarrow \quad \hat{x}_1 = \langle 1, 0, 0 \rangle, \quad \hat{x}_2 = \langle 0, 1, 0 \rangle, \quad \hat{x}_3 = \langle 0, 0, 1 \rangle$$

$$\text{Def}^{n}/ \quad \vec{A}\cdot\vec{B} = \sum_{i=1}^{3} A_i B_i$$

$$\vec{A}\times\vec{B} = \sum_{i,j,k} \epsilon_{ijk} A_i B_j \hat{x}_k$$

Here any vector has a decomposition into vector components

$$\begin{aligned}
\vec{A} &= \langle A_1, A_2, A_3 \rangle \\
&= \langle A_1, 0, 0\rangle + \langle 0, A_2, 0\rangle + \langle 0, 0, A_3\rangle \\
&= A_1 \langle 1,0,0\rangle + A_2\langle 0,1,0\rangle + A_3\langle 0,0,1\rangle \\
&= A_1 \hat{x}_1 + A_2 \hat{y} + A_3 \hat{z}
\end{aligned}$$

Notice $A_i = \vec{A}\cdot\hat{x}_i$, consider $\vec{A} = \sum_{j=1}^{3} A_j \hat{x}_j$

$$\begin{aligned}
\vec{A}\cdot\hat{x}_i &= \left(\sum_{j=1}^{3} A_j \hat{x}_j\right)\cdot\hat{x}_i \\
&= \sum_{j=1}^{3} A_j\, \hat{x}_j\cdot\hat{x}_i \\
&= \sum_{j=1}^{3} A_j \delta_{ji} \\
&= A_i
\end{aligned}$$

Observation

$$\hat{x}_i\cdot\hat{x}_j = \delta_{ij}$$

Let us review basic properties of the dot and cross-product,

(1.) $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$ and $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$

(2.) $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}$ and $\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B}+\vec{A}\times\vec{C}$

(3.) $\vec{A}\cdot(c\vec{B}) = c(\vec{A}\cdot\vec{B})$ and $\vec{A}\times(c\vec{B}) = c\,\vec{A}\times\vec{B}$

(4.) $\vec{A}\cdot\vec{A} \geq 0$ whereas $\vec{A}\times\vec{A} = 0$.

(5.) $\vec{A}\cdot\vec{B} = AB\cos\theta$ and $\|\vec{A}\times\vec{B}\| = AB\sin\theta$

I will not attempt to be comprehensive, of cource we also know $(\vec{A}+\vec{B})\cdot\vec{C} = \vec{A}\cdot\vec{C}+\vec{B}\cdot\vec{C}$ etc. It is instructive to study proofs of (1) — (5) to illustrate the beauty of the index notation.

(1.) $\vec{A}\cdot\vec{B} = \sum_i A_iB_i = \sum_i B_iA_i = \vec{B}\cdot\vec{A}$

(1.) $$(\vec{A}\times\vec{B})_k = \sum_{i,j} \epsilon_{ijk}A_iB_j$$

$$= \sum_{j,i} -\epsilon_{jik}B_jA_i$$

$$= -(\vec{B}\times\vec{A})_k \quad \therefore \underline{\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}}.//$$

(2.) $\vec{A}\cdot(\vec{B}+\vec{C}) = \sum_i A_i (\vec{B}+\vec{C})_i$

$$= \sum_i A_i (B_i + C_i)$$

$$= \sum_i (A_i B_i + A_i C_i)$$

$$= \sum_i A_i B_i + \sum_i A_i C_i$$

$$= \underline{\vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}} \,.//$$

and,

$$\vec{A}\times(\vec{B}+\vec{C}) = \sum_{i,j,k} \epsilon_{ijk} A_i (B_j + C_j)\hat{x}_k$$

$$= \sum_{i,j,k} \epsilon_{ijk} A_i B_j \hat{x}_k + \sum_{i,j,k} \epsilon_{ijk} A_i C_j \hat{x}_k$$

$$= \underline{\vec{A}\times\vec{B} + \vec{A}\times\vec{C}} \,.//$$

(3.) $\vec{A}\cdot(c\vec{B}) = \sum_j A_j (c\vec{B})_j$

$$= \sum_j A_j c B_j$$

$$= c\sum_j A_j B_j$$

$$= \underline{c\,\vec{A}\cdot\vec{B}} \,.//$$

$$\vec{A}\times(c\vec{B}) = \sum_{i,j,k} \epsilon_{ijk} A_i (cB_j)\hat{x}_k$$

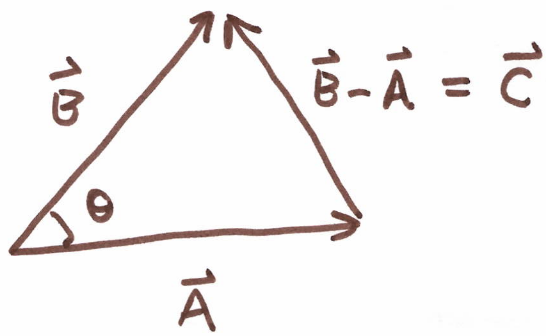
$$= c\sum_{i,j,k} \epsilon_{ijk} A_i B_j \hat{x}_k$$

$$= \underline{c\,\vec{A}\times\vec{B}} \,.//$$

(4.) $\vec{A}\cdot\vec{A} = \sum_i A_i A_i = \sum_{i=1}^{3} (A_i)^2 \geq 0 \,.//$

$\vec{A}\times\vec{A} = -\vec{A}\times\vec{A} \Rightarrow 2\vec{A}\times\vec{A} = 0 \;\therefore\; \underline{\vec{A}\times\vec{A} = 0} \,.//$

(5.)



$$\vec{C}\cdot\vec{C} = (\vec{B}-\vec{A})\cdot(\vec{B}-\vec{A}) = \vec{B}\cdot\vec{B} - \vec{B}\cdot\vec{A} - \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{A}$$

$$\Rightarrow C^2 = A^2 + B^2 - 2\vec{A}\cdot\vec{B} \underset{\text{Law of Cosines}}{=} A^2 + B^2 - 2AB\cos\theta$$

$$\therefore \quad \underline{\vec{A}\cdot\vec{B} = AB\cos\theta} \;.//$$

Remark: the proof that $\|\vec{A}\times\vec{B}\| = AB\sin\theta$ requires a deeper magic.

(6.) $\underbrace{\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})}_{\text{cyclicity of the triple product}}$

(7.) $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$
$= (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$

(8.) $\vec{A}\times(\vec{B}\times\vec{C}) + \vec{B}\times(\vec{C}\times\vec{A}) + \vec{C}\times(\vec{A}\times\vec{B}) = 0$

Proofs of (7.) and (8.) I've gifted you in homework, we discuss (6) ↘

(6.) derivation of cyclicity of $\vec{A}\cdot(\vec{B}\times\vec{C})$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \sum_k A_k (\vec{B}\times\vec{C})_k$$

$$= \sum_k A_k \sum_{i,j} \epsilon_{ijk} B_i C_j \quad (*)$$

$$= \sum_i \sum_{j,k} B_i \epsilon_{ijk} A_k C_j$$

$$= \sum_i B_i \sum_{j,k} \epsilon_{ijk} C_j A_k \quad \text{(flip } i,j\text{)}$$

$\epsilon_{ijk} \rightarrow -\epsilon_{jik}$ (flip i,k)

$$= \sum_i B_i \sum_{j,k} \epsilon_{jki} C_j A_k$$

$$= \sum_i B_i (\vec{C}\times\vec{A})_i$$

$$= \vec{B}\cdot(\vec{C}\times\vec{A})$$

$$= \sum_j C_j \sum_{k,i} \epsilon_{ijk} A_k B_i \quad \leftarrow \text{from } (*)$$

$\downarrow **$ $\quad \epsilon_{ijk} = -\epsilon_{ikj} = \epsilon_{kij}$

$$= \sum_j C_j \sum_{k,i} \epsilon_{kij} A_k B_i$$

$$= \sum_j C_j (\vec{A}\times\vec{B})_j$$

$$= \vec{C}\cdot(\vec{A}\times\vec{B}) \,//$$

Remark: if you know $\vec{A}\cdot(\vec{B}\times\vec{C}) = \det\left[\vec{A}\,|\,\vec{B}\,|\,\vec{C}\right]$ that's another way.

Remark: (This is a digression, I don't think we need this, but it's cool,

$$\det(A) = \sum_{i_1, i_2, \dots, i_n = 1}^{n} \epsilon_{i_1 i_2 \cdots i_n} A_{i_1 1} A_{i_2 2} \cdots A_{i_n n}$$

is a def$^{\underline{n}}$ for determinant of $n \times n$ matrix where $\epsilon_{123\dots n} = 1$ and all other values are generated by complete antisymmetry.

Hence,

$$\det(\vec{A}|\vec{B}|\vec{C}) = \sum_{i,j,k} \epsilon_{ijk} A_i B_j C_k$$
$$= \sum_k C_k \sum_{i,j} \epsilon_{ijk} A_i B_j$$
$$= \vec{C} \cdot (\vec{A} \times \vec{B}) \text{ etc.}$$

oh, enough.

Following Griffiths pg. 8, eq$^{\underline{n}}$ 1.18, let's try to derive it

$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = \sum_k (\vec{A} \times \vec{B})_k (\vec{C} \times \vec{D})_k$$
$$= \sum_k \sum_{i,j} \epsilon_{ijk} A_i B_j \sum_{l,m} \epsilon_{lmk} C_l D_m$$
$$= \sum_{i,j,k,l,m} \underbrace{\epsilon_{ijk} \epsilon_{lmk}}_{\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}} A_i B_j C_l D_m$$

Continuing, we use the nontrivial ϵ - contraction identity from last page,

$$(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = \sum_{i,j,\ell,m}(\delta_{i\ell}\delta_{jm}-\delta_{im}\delta_{j\ell})A_i B_j C_\ell D_m$$

$$= \sum_{i,j,\ell,m}(\delta_{i\ell}\delta_{jm}A_i B_j C_\ell D_m - \delta_{im}\delta_{j\ell}A_i B_j C_\ell D_m)$$

$$= \sum_{i,j} A_i B_j C_i D_j - \sum_{i,j} A_i B_j C_j D_i$$

$$= \left(\sum_i A_i C_i\right)\left(\sum_j B_j D_j\right) - \left(\sum_i A_i D_i\right)\left(\sum_j B_j C_j\right)$$

$$= (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})$$

This is what lurks behind (1.18) in Griffiths. It is the deeper magic I mentioned,

$$(S.)\quad (\vec{A}\times\vec{B})\cdot(\vec{A}\times\vec{B}) = (\vec{A}\cdot\vec{A})(\vec{B}\cdot\vec{B}) - (\vec{A}\cdot\vec{B})(\vec{B}\cdot\vec{A})$$

$$= A^2B^2 - (AB\cos\theta)^2$$

$$= A^2B^2(1-\cos^2\theta)$$

$$= A^2B^2\sin^2\theta$$

$$\therefore\ \underline{\|\vec{A}\times\vec{B}\| = AB\sin\theta}\ //$$

not obvious. But, nice to know!

We work with right-handed coordinate systems almost w/o exception. The basic idea is the unit-vectors for such a coord. system must form a <u>right</u>-<u>handed</u> <u>frame</u> (or triple)

Defn/ Given <u>unit</u>-vectors $\hat{u}_1, \hat{u}_2, \hat{u}_3$ we say $\{\hat{u}_1, \hat{u}_2, \hat{u}_3\}$ is a right-handed-frame if $\underbrace{\hat{u}_1 \times \hat{u}_2 = \hat{u}_3}_{\text{I}}$ and $\underbrace{\hat{u}_2 \times \hat{u}_3 = \hat{u}_1}_{\text{II}}$ and $\underbrace{\hat{u}_3 \times \hat{u}_1 = \hat{u}_2}_{\text{III}}$

Notice we may concisely express all three conditions via

$$\boxed{\hat{u}_i \times \hat{u}_j = \sum_k \epsilon_{ijk} \hat{u}_k}$$

Obvious and quintessiential example,

$$\hat{x} \times \hat{y} = \hat{z}$$
$$\hat{y} \times \hat{z} = \hat{x}$$
$$\hat{z} \times \hat{x} = \hat{y}$$

We saw in lecture that any one of I, II or III implies the remaining pair hold true. I'll replicate that argument ↷

Lemma: given unit-vectors $\hat{u}_1, \hat{u}_2, \hat{u}_3$ for which $\hat{u}_i \cdot \hat{u}_j = \delta_{ij}$ and any one of

I $\hat{u}_1 \times \hat{u}_2 = \hat{u}_3$

II $\hat{u}_2 \times \hat{u}_3 = \hat{u}_1$

III $\hat{u}_3 \times \hat{u}_1 = \hat{u}_2$

hold then all of I, II and III hold hence $\{\hat{u}_1, \hat{u}_2, \hat{u}_3\}$ is right-handed frame.

Consider the case we're given $\hat{u}_1 \times \hat{u}_2 = \hat{u}_3$,

$$\hat{u}_2 \times \hat{u}_3 = \left[(\hat{u}_2 \times \hat{u}_3) \cdot \hat{u}_1\right]\hat{u}_1 + \underbrace{\left[(\hat{u}_2 \times \hat{u}_3) \cdot \hat{u}_2\right]\hat{u}_2 + \left[(\hat{u}_2 \times \hat{u}_3) \cdot \hat{u}_3\right]\hat{u}_3}$$

$$= \left((\hat{u}_1 \times \hat{u}_2) \cdot \hat{u}_3\right)\hat{u}_1$$

$$= (\hat{u}_3 \cdot \hat{u}_3)\hat{u}_1$$

$$= \hat{u}_1$$

work out to zero by cyclicity of triple prod. and $\hat{u}_2 \times \hat{u}_2 = 0$ and $\hat{u}_3 \times \hat{u}_3 = 0$

Similar calculation shows $\hat{u}_3 \times \hat{u}_1 = \hat{u}_2$, and further similar calculations hold if we're given II or III.

Remark: I'm assuming you know three $\perp$ unit-vectors in $\mathbb{R}^3$ can be used to build any vector in

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continued, $\mathbb{R}^3$. Moreover, for $\hat{u}_i \cdot \hat{u}_j = \delta_{ij}$ we have $\vec{A} = \sum_{i=1}^{3} (\vec{A} \cdot \hat{u}_i)\hat{u}_i$. Of course the most familiar case is

$$\begin{aligned}
\vec{A} &= \langle A_1, A_2, A_3 \rangle \\
&= \langle \vec{A}\cdot\hat{x}_1, \vec{A}\cdot\hat{x}_2, \vec{A}\cdot\hat{x}_3 \rangle \\
&= (\vec{A}\cdot\hat{x}_1)\hat{x}_1 + (\vec{A}\cdot\hat{x}_2)\hat{x}_2 + (\vec{A}\cdot\hat{x}_3)\hat{x}_3 \\
&= \sum_{i=1}^{3} (\vec{A}\cdot\hat{x}_i)\hat{x}_i
\end{aligned}$$

(same pattern holds for $\hat{u}_1, \hat{u}_2, \hat{u}_3$ given $\hat{u}_i \cdot \hat{u}_j = \delta_{ij}$.)

Proposition: if $\vec{u}_1, \vec{u}_2, \vec{u}_3$ are nonzero and satisfy $\vec{u}_1 \times \vec{u}_2 = \vec{u}_3$, $\vec{u}_2 \times \vec{u}_3 = \vec{u}_1$, $\vec{u}_3 \times \vec{u}_1 = \vec{u}_2$ then in fact $\vec{u}_i \cdot \vec{u}_j = \delta_{ij}$ hence $\vec{u}_i = \hat{u}_i$ for $i = 1, 2, 3$.

Notice
$$\begin{aligned}
\vec{u}_1 \cdot \vec{u}_2 &= (\vec{u}_2 \times \vec{u}_3) \cdot \vec{u}_2 = 0 \\
\vec{u}_1 \cdot \vec{u}_3 &= (\vec{u}_2 \times \vec{u}_3) \cdot \vec{u}_3 = 0 \\
\vec{u}_2 \cdot \vec{u}_3 &= (\vec{u}_3 \times \vec{u}_1) \cdot \vec{u}_3 = 0
\end{aligned}$$

Likewise, $\vec{u}_1 \cdot \vec{u}_1 = (\vec{u}_2 \times \vec{u}_3)\cdot(\vec{u}_2 \times \vec{u}_3) = (\vec{u}_2 \cdot \vec{u}_2)(\vec{u}_3 \cdot \vec{u}_3) - 0$.

thus $\|\vec{u}_1\|^2 = \|\vec{u}_2\|^2 \|\vec{u}_3\|^2$ and similarly $\|\vec{u}_2\|^2 = \|\vec{u}_1\|^2 \|\vec{u}_3\|^2$ and $\|\vec{u}_3\|^2 = \|\vec{u}_1\|^2 \|\vec{u}_2\|^2 \Rightarrow \|\vec{u}_i\| = 1$ for all i.//

Gradient, Divergence and Curl

To begin we focus on the Cartesian formulation. We'll discuss the geometric meaning later.

Def^{n}/ $\nabla = \hat{x}\dfrac{\partial}{\partial x} + \hat{y}\dfrac{\partial}{\partial y} + \hat{z}\dfrac{\partial}{\partial z} = \sum_{i=1}^{3} \hat{x}_i \dfrac{\partial}{\partial x_i}$

$$\nabla f = \sum_{i=1}^{3} \hat{x}_i \frac{\partial f}{\partial x_i} = \sum_{i=1}^{3} (\partial_i f)\hat{x}_i$$

$$\nabla \cdot \vec{G} = \sum_{i=1}^{3} \frac{\partial G_i}{\partial x_i}$$

$$\nabla \times \vec{H} = \sum_{i,j,k} \epsilon_{ijk} \frac{\partial F_j}{\partial x_i} \hat{x}_k = \sum_{i,j,k} \epsilon_{ijk} (\partial_i F_j)\hat{x}_k$$

These derivatives enjoy natural linearity and product rules (see hwk for prod & quotient rule derivations)

(1.) $\nabla(cf + g) = c\nabla f + \nabla g$

$\nabla(fg) = (\nabla f)g + f\nabla g$

(2.) $\nabla \cdot (f\vec{G}) = (\nabla f)\cdot \vec{G} + f\,\nabla \cdot \vec{G}$

$\nabla \cdot (c\vec{G} + \vec{H}) = c\nabla \cdot \vec{G} + \nabla \cdot \vec{H}$

(3.) $\nabla \times (c\vec{G} + \vec{H}) = c\,\nabla \times \vec{G} + \nabla \times \vec{H}$

$\nabla \times (f\vec{G}) = (\nabla f)\times G + f\,\nabla \times \vec{G}$

Proof of (1), (2), (3) rests on linearity of $\dfrac{\partial}{\partial x_i}$ and its prod. rule ↘

(1.) $$\nabla(cf+g) = \sum_{i=1}^{3} \hat{x}_i \frac{\partial}{\partial x_i}(cf+g)$$

$$= \sum_{i=1}^{3} \hat{x}_i \left(c\frac{\partial f}{\partial x_i} + \frac{\partial g}{\partial x_i} \right)$$

$$= c\sum_{i=1}^{3} \hat{x}_i \frac{\partial f}{\partial x_i} + \sum_{i} \hat{x}_i \frac{\partial g}{\partial x_i}$$

$$= \underline{c\nabla f + \nabla g} \; //$$

$$\nabla(fg) = \sum_{i=1}^{3} \hat{x}_i \frac{\partial}{\partial x_i}(fg)$$

$$= \sum_{i=1}^{3} \hat{x}_i \left(\frac{\partial f}{\partial x_i} g + f\frac{\partial g}{\partial x_i} \right)$$

$$= \left(\sum_{i=1}^{3} \hat{x}_i \frac{\partial f}{\partial x_i} \right) g + f\left(\sum_{i=1}^{3} \hat{x}_i \frac{\partial g}{\partial x_i} \right)$$

$$= \underline{(\nabla f)g + f(\nabla g)} \; //$$

Remark: proofs of (2.) and (3.) are similar, but require attention to details of def$^{\underline{n}}$ of $\nabla \cdot \vec{G}$ and $\nabla \times \vec{H}$. I'll leave the quotient rules for hwk to discuss.

Geometric Interpretations for ∇

- The gradient is simplest, given function f we find $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$ points in the direction for which f increases fastest.

$$\frac{d}{dt}\big(f(\vec{r}(t))\big) = \underbrace{\nabla f(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}}_{\text{this is largest}}$$

when $\frac{d\vec{r}}{dt} \parallel \nabla f(\vec{r}(t))$

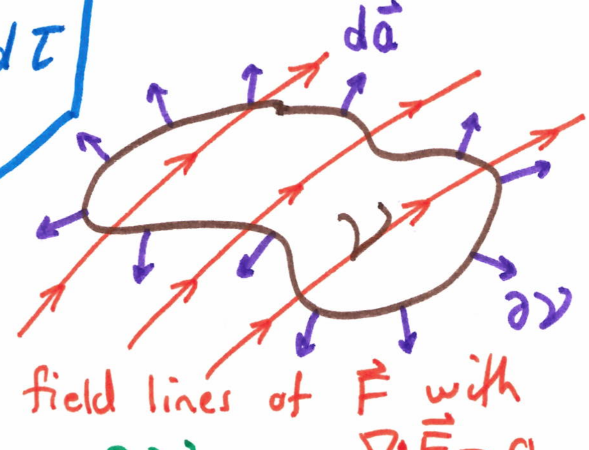
(follow ∇f if you wish to level-up f)

- The divergence detects the creation of new field lines for a given vector field.

$$\nabla \cdot \vec{F} \neq 0 \Rightarrow \text{new field lines for } \vec{F} \text{ at such points}$$

Divergence Th^m

$$\boxed{\underbrace{\int_{\partial \mathcal{V}} \vec{F} \cdot d\vec{a}}_{\text{flux of } \vec{F} \text{ through } \partial\mathcal{V}} = \int_{\mathcal{V}} (\nabla \cdot \vec{F})\, d\tau}$$



Def^n/ volume $\mathcal{V}$ has $\underbrace{\text{boundary } \partial\mathcal{V}}_{\text{positively oriented.}}$

We can calculate $\nabla \cdot \vec{F}$ at a point P by calculating flux of $\vec{F}$ through small bubble about point P

$$\int_{S_R} \vec{F} \cdot d\vec{a} = \int_{B_R} (\nabla \cdot \vec{F}) \, d\tau$$

S_R: spherical shell of radius R centered at P

B_R: ball of radius R centered at P

$\partial B_R = S_R$

$$\lim_{R \to 0} \int_{S_R} \vec{F} \cdot d\vec{a} = \lim_{R \to 0} \int_{B_R} (\nabla \cdot \vec{F}) \, d\tau$$

$$\int_{S_R} \vec{F} \cdot d\vec{a} \cong (\nabla \cdot \vec{F}) \int_{B_R} d\tau = \left(\tfrac{4}{3}\pi R^3\right) \nabla \cdot \vec{F}$$

$$\boxed{\nabla \cdot \vec{F} = \lim_{R \to 0} \frac{3}{4\pi R^3} \int_{S_R} \vec{F} \cdot d\vec{a}}$$

Example: calculation of divergence via $\int$

$$\vec{F} = \langle x, y, z\rangle = \vec{r} = r\hat{r}$$

$$\nabla\cdot\vec{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = \boxed{3}$$

$$\int_{S_R}\vec{F}\cdot d\vec{a} = \int_{S_R}\vec{r}\cdot\left(r^2\sin\theta\, d\theta\, d\phi\right)\hat{r}$$

$$= \int_0^{\pi}\int_0^{2\pi} R^3\sin\theta\, d\theta\, d\phi$$

$$= R^3\int_0^{\pi}\sin\theta\, d\theta\int_0^{2\pi} d\phi$$

$$= 4\pi R^3$$

woo hoo.

Then,

$$\lim_{R\to 0}\left(\frac{3}{4\pi R^3}\int_{S_R}\vec{F}\cdot d\vec{a}\right) = \lim_{R\to 0}\left(\frac{3}{4\pi R^3}\cdot 4\pi R^3\right) = \boxed{3}$$

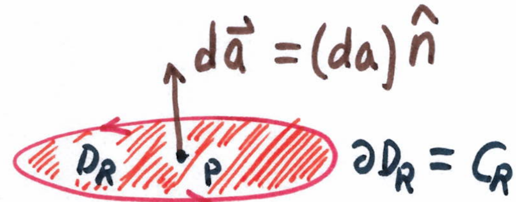
Remark: this limiting procedure seems like a very hard way to calculate the divergence. The interesting thing is it gives us a way to <u>derive</u> formulas for divergence in non-Cartesian coordinate systems (Appendix A of Griffiths)

• CURL OF $\vec{F}$ has meaning made clear from Stokes' Th$^{\underline{m}}$:

$$\int_{\partial S} \vec{F} \cdot d\vec{l} = \int_{S} (\nabla \times \vec{F}) \cdot d\vec{a}$$

the circulation of $\vec{F}$ along ∂S should be zero if $\nabla \times \vec{F}$ is everywhere 0 on S. Of course there are other ways the flux of $\nabla \times \vec{F}$ could be zero through S. If we consider a limiting procedure then we'll find a way to calculate $\nabla \times \vec{F}$ from $\vec{F}$'s circulation about a little loop, to find $\nabla \times \vec{F}$ at P,

$$\int_{C_R} \vec{F} \cdot d\vec{l} = \int_{D_R} (\nabla \times \vec{F}) \cdot d\vec{a} \qquad (\partial D_R = C_R)$$



should pick up on $\hat{n}$-component of $\nabla \times \vec{F}$

<u>As $R \to 0$</u>,

$$\int_{C_R} \vec{F} \cdot d\vec{l} = \int_{D_R} (\nabla \times \vec{F}) \cdot (da)\hat{n} = (\nabla \times \vec{F}) \cdot \hat{n} \int_{D_R} da$$

$$\boxed{(\nabla \times \vec{F}) \cdot \hat{n} = \lim_{R \to 0} \left(\frac{1}{\pi R^2} \int_{C_R} \vec{F} \cdot d\vec{l} \right)}$$