LECTURE 1: WHAT IS TOPOLOGY?

- I'm following Marco Manetti's excellent text titled TOPOLOGY.
  - TOPOLOGY IS ABSTRACT ANALYSIS.
- Let's informally review analysis on $\mathbb{R}$ or $\mathbb{R}^n$

$$d(x,y) = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\dots+(x_n-y_n)^2} = \|x-y\| = \sqrt{(x-y)\cdot(x-y)}$$

distance between x and y

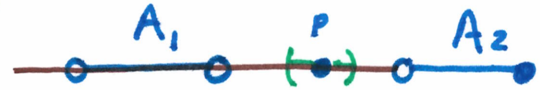
Def^n/ A point $x \in \mathbb{R}^n$ is called adherent to a subset $A \subseteq \mathbb{R}^n$ if it is possible to find points of A that lie arbitrarily close to x.

(x adheres to A iff for any $\delta > 0$ there exists $p \in A$ such that $d(p,x) < \delta$)

E1

E2

E3

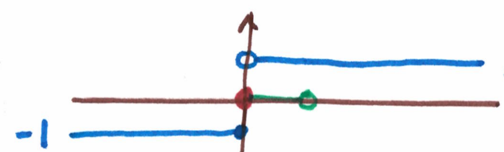


$A = A_1 \cup \{p\} \cup A_2$

$\underbrace{\text{"A is not continuous"}}_{\text{antiquated language}}$

Def^n/ Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$. A map $f : X \longrightarrow Y$ is called continuous if for any subset $A \subset X$ and point $x \in X$ adherent to A the point $f(x)$ adheres to $f(A)$ ← continuity of a map.

E4 $f(x) = \dfrac{x}{|x|}$ for $x \neq 0$



$f(0) = -1$ then $f : \mathbb{R} \to \mathbb{R}$ has $f(0,1) = \{1\}$ and 0 adheres to $(0,1)$ but $f(0) = -1$.

PROPERTIES OF CONTINUOUS MAPS ON EUCLIDEAN SPACES

Let X, Y, Z be subsets of Euclidean spaces $\qquad f|_W : W \to Y$

(C1) If $f: X \to Y$ cont. and $W \subset X$, $Z \subset Y$ such that $f(W) \subset Z$ then $f: W \to Z$ cont.

(C2) If $f: X \to Y$ and $g: Y \to Z$ cont. then $g \circ f: X \to Z$ is cont.

(C3) If $X \subset Y$ then the inclusion map $i: X \to Y$ is cont.

(C4) Let $f_j : X \to \mathbb{R}$ $j = 1, 2, .., n$ denote component functions of $f: X \to \mathbb{R}^n$ meaning $f(x) = (f_1(x), f_2(x), .., f_n(x))$ $\forall x \in X$. Then f cont. iff f_j cont. $\forall j = 1, 2, .., n$

(C5) The following maps are continuous

1.) every linear map $\mathbb{R}^n \to \mathbb{R}$
2.) multiplication $\mathbb{R}^2 \to \mathbb{R}$, $(x, y) \longmapsto xy$
3.) inversion $\mathbb{R} - \{0\} \to \mathbb{R} - \{0\}$ $x \longmapsto x^{-1}$
4.) exponential function, natural log, sine, cosine and all elementary functions on their domains
5.) absolute value $\mathbb{R} \to \mathbb{R}$, $x \longmapsto |x|$
6.) max, min $(x, y) \longmapsto \max(x, y)$ $(x, y) \longmapsto \min(x, y)$.

Remark: all these claims can be proved by sorting through ε, δ arguments. I've done much of this in an advanced calculus course from Edwards

$\boxed{E5}$ Given f, g continuous can argue $(f, g): X \to \mathbb{R}^2$ continuous and $(x, y) \underset{\text{add}}{\longmapsto} x + y$, $(x, y) \underset{\text{multiply}}{\longmapsto} xy$ cont. $\Rightarrow$ $\underbrace{f + g = \text{add} \circ (f, g) \quad fg = \text{multiply} \circ (f, g)}_{\text{continuous}}$

Def^{n}/ A subset $C \subset X$ in $\mathbb{R}^n$ is called <u>closed</u> in X if it coincides with the set of points of X that adhere to C. Equivalently, C is closed in X if for any $x \in X - C = \{x \in X \mid x \notin C\}$ a number $\delta > 0$ exists such that $d(x,y) \geq \delta$ for every $y \in C$

← Closed Set in terms of adherence

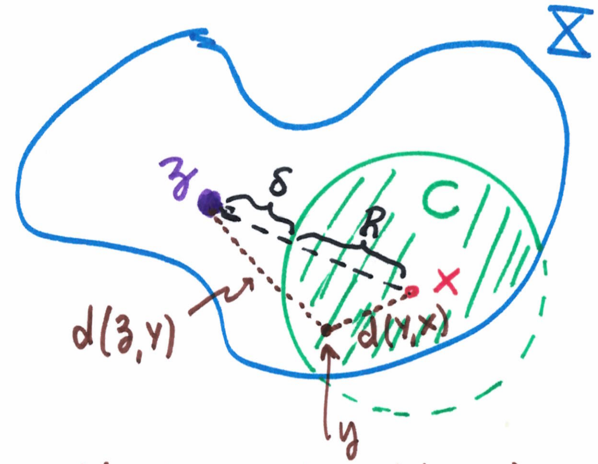
$\boxed{E6}$ Let $X \subset \mathbb{R}^n$ and $R > 0$ and fix $x \in \mathbb{R}^n$. Let

$$C = \{ y \in X \mid d(x,y) \leq R \}.$$

We show C is closed inside X, If $z \in X - C$ then let $\delta = d(x,z) - R > 0$. Therefore,

$$d(z,y) \geq d(x,z) - d(y,x) \geq \delta$$

for every $y \in C$ $\therefore$ z is not adherent to C.



$d(y,x) + d(z,y) \geq d(x,z)$
(triangle inequality)

$\boxed{E7}$ $X \subseteq \mathbb{R}^n$ and $f : X \to \mathbb{R}$ cont.

Then $C = \{x \in X \mid f(x) = 0\}$ is closed in X.

<u>Why?</u> if $x \in X$ is adherent to C then $f(x)$ is adherent to $\{0\}$ $\therefore$ $f(x) = 0$.

Thus $x \in C$ and we find C is closed. Similary, if $Z \subset \mathbb{R}$ closed then $\{x \in X \mid f(x) \in Z\}$ is closed in X.

<u>Th^{m}(1.8)</u>/ Glueing Lemma if $X = A \cup B$ (both closed) and $f : X \to Y$ is fnct. with $f|_A$ and $f|_B$ cont. then f cont.

EXAMPLES OF HOMEOMORPHISMS

Def^n/ A homeomorphism is a continuous and bijective map with continuous inverse. Two subsets of $\mathbb{R}^n$ are called homeomorphic if $\exists$ homeomorphism mapping one of the subsets to the other.

E8

$]0,1[= (0,1)$

$]0,2[= (0,2)$

$]0,\infty[= (0,\infty)$

$]-\infty,\infty[= (-\infty,\infty) = \mathbb{R}$

$f(x) = e^x$ has $f(-\infty,\infty) = (0,\infty)$

$g(x) = e^{-x}$ has $g(0,\infty) = (0,1)$

$h(x) = 2x$ has $h(0,1) = (0,2)$

E9

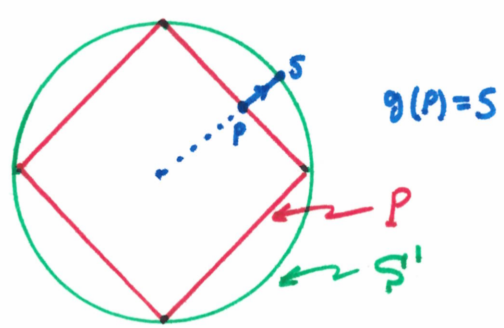
$S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}$

$P = \{(x,y) \in \mathbb{R}^2 \mid |x|+|y| = 1$

$f: S^1 \to P \qquad f(x,y) = \left(\frac{x}{|x|+|y|}, \frac{y}{|x|+|y|}\right)$

$g: P \to S^1 \qquad g(x,y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$

$g = f^{-1}$ and these are both continuous.



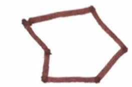
for future reference.

Def^n/ $D^n = \{x \in \mathbb{R}^n \mid \|x\| \le 1\}$ the closed unit ball of dimension n

$S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ the unit n-sphere

$B(x,R) = \{y \in \mathbb{R}^n \mid d(x,y) < R\}$ the open ball centered at x with radius R.

$\boxed{E10}$ An <u>affine transformation</u> of $\mathbb{R}^2$ has form $f(x) = Ax + b$ where $\det(A) \neq 0$ and $A \in \mathbb{R}^{2\times 2}$. These are homeomorphisms. Given two triangles T_1 and T_2 $\exists\, f: T_1 \longrightarrow T_2$ where f affine. Likewise for n-sided shapes (no pinching)



ok

not ok

$\boxed{E11}$ $Q = \{ (x,y) \mid |x| + |y| \leq 1 \}$ ← square (filled in)

$T = \{ (x,y) \in Q \mid y \leq 0 \}$ ← triangle (filled in)

$f: Q \longrightarrow T$ defined by $f(x,y) = \left(x,\ \frac{1}{2}(y + |x| - 1)\right)$ is homeomorphism

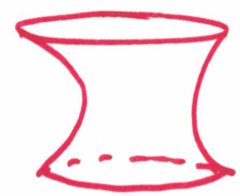
$\boxed{E12}$ $X = \mathbb{R}^2 - \{(0,0)\}$

punctured plane

$Y = \{ (x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1 \}$

circular cylinder

$Z = \{ (x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 1 \}$



1-sheet hyperboloid

$f: Y \longrightarrow X$ by $\underbrace{f(x,y,z) = (xe^z, ye^z)}$

it's slicing Y into circles which it
stretches or compresses by e^z to fill
circle of radius $e^z \neq 0$ in X
Lower half of Y maps inside D^2
Upper half maps to exterior of D^2 in $\mathbb{R}^2$

$g: Z \rightarrow Y$ by $g(x,y,z) = \left(\frac{x}{\sqrt{1+z^2}},\ \frac{y}{\sqrt{1+z^2}},\ z \right)$ (cool)

$\boxed{E13}$ $\underbrace{f(x) = Rx + P}_{R>0,\ x, P \in \mathbb{R}^n}$ gives homeomorphism of $\underbrace{B(0,1) \text{ and } B(P,R)}_{\text{open balls homeomorphic.}}$

Likewise $\underbrace{g(x) = \dfrac{x}{\sqrt{1+\|x\|^2}}}_{\text{this shrinks } \mathbb{R}^n \text{ down into } B(0,1)}$ defines homeomorphism $g : \mathbb{R}^n \longrightarrow B(0,1)$

$\boxed{E14}$ $S^n = \{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \|x\|^2 = 1 \}$

$N = (1, 0, \dots, 0)$ be NORTH POLE

(see fig. 1.9 and Manetti for more details here)

$f : S^n - \{N\} \longrightarrow \mathbb{R}^n$

$$f(x_0, \dots, x_n) = \frac{1}{1-x_0}(x_1, \dots, x_n)$$

$$f^{-1}(y_1, \dots, y_n) = \left(\frac{\|y\|^2 - 1}{1+\|y\|^2}, \frac{2y_1}{1+\|y\|^2}, \dots, \frac{2y_n}{1+\|y\|^2} \right)$$ where $\|y\|^2 = y_1^2 + y_2^2 + \dots + y_n^2$

$\boxed{E15}$ $r : \mathbb{R}^3 - \{0\} \longrightarrow \mathbb{R}^3 - \{0\}$ by $r(x) = \dfrac{x}{\|x\|^2}$ is a bijection whose inverse is itself.

Notice, $r(r(x)) = r\left(\dfrac{x}{\|x\|^2}\right) = \dfrac{x/\|x\|^2}{\| x/\|x\|^2 \|^2} = \dfrac{x}{\|x\|^2 \left\| \frac{x}{\|x\|^2} \right\|^2} = \dfrac{x}{\frac{\|x\|^2}{\|x\|^4} \|x\|^2} = x.$

- complement of circle in $\mathbb{R}^3$ is homeomorphic to complement of line and a pt.

$K = \{ x \in \mathbb{R}^3 \mid x_3 = 0, (x_1 - 1)^2 + x_2^2 + \cancel{x_3^2} = 1 \}$ (see Manetti p. 14 for the rest of this)

E16

$$\underbrace{SU(2,\mathbb{C})}_{\text{special unitary matrices}} = \{ M \in \mathbb{C}^{2\times 2} \mid \underbrace{(\overline{M})^T M}_{M^\dagger = (\overline{M})^T \text{ (Hermitian Conjugate)}} = I \;,\; \det(M) = 1 \}$$

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{with} \quad \underbrace{MM^\dagger = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}}= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad ad - bc = 1$$

$$a\bar{a} + b\bar{b} = 1 \quad \therefore \quad |a|^2 + |b|^2 = 1$$

$$c\bar{a} + d\bar{b} = 0$$

$$a\bar{c} + b\bar{d} = 0$$

$$c\bar{c} + d\bar{d} = 1 \quad \therefore \quad |c|^2 + |d|^2 = 1$$

$$1 = ad - bc = |a|^2 + |b|^2 = |c|^2 + |d|^2 \quad \text{and} \quad \underbrace{\bar{a}c + d\bar{b}} = \bar{c}a + \bar{d}b = 0$$

Likewise, $d = \bar{a}$

$$a\bar{a}c + ad\bar{b} = 0$$

$$a\bar{a}c + (1+bc)\bar{b} = 0$$

$$(|a|^2 + |b|^2)c + \bar{b} = 0$$

$$\therefore \quad \underline{c = -\bar{b}}.$$

$$M \in SU(2,\mathbb{C}) \implies M = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$$

with $a\bar{a} + b\bar{b} = \underbrace{|a|^2 + |b|^2}_{a_1^2 + a_2^2 + b_1^2 + b_2^2 = 1} = 1$ $\quad \begin{pmatrix} a = a_1 + ia_2 \in \mathbb{C} \\ b = b_1 + ib_2 \in \mathbb{C} \end{pmatrix}$

$f\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} = (a_1, a_2, b_1, b_2)$ defines $f : SU(2,\mathbb{C}) \longrightarrow S^3$ homeomorphism

NOTE: conclude with comments on p. 16 of Manetti on finite # of pts. example etc...