LECTURE 20: BOUNDARY CONDITIONS & MATTER WITH MAGNETOSTATICS

There are a few odds and ends I should have covered already. ① the plausibility of the COULOMB GAUGE and ② BOUNDARY CONDITIONS FOR $\vec{B}$ and $\vec{A}$.

CLAIM: it is possible to choose $\vec{A}$ for which $\vec{B} = \nabla \times \vec{A}$ and $\nabla \cdot \vec{A} = 0$ (this is the COULOMB GAUGE)

Let $\vec{A}_o$ be a vector potential with $\vec{B} = \nabla \times \vec{A}_o$.
Construct $\vec{A} = \vec{A}_o + \nabla\lambda$ then $\nabla \times \vec{A} = \nabla \times (\vec{A}_o + \nabla\lambda) = \nabla \times \vec{A}_o = \vec{B}$ hence $\vec{A}$ is a vector potential for $\vec{B}$. Notice,

$$\nabla \cdot \vec{A} = \nabla \cdot \vec{A}_o + \nabla \cdot (\nabla \lambda)$$

Thus $\nabla \cdot \vec{A} = 0$ provided we select λ for which

$$\nabla^2 \lambda = -\nabla \cdot \vec{A}_o$$

But this is just Poisson's Eq$^{\underline{n}}$ where $\nabla^2 V = -\rho/\epsilon_o$ has $\nabla \cdot \vec{A}_o = \rho/\epsilon_o$ hence we know how to find such λ. For example, if $\nabla \cdot \vec{A}_o = 0$ for $r >> 0$ then we can construct $\lambda(r) = \dfrac{1}{4\pi}\displaystyle\int \frac{\nabla \cdot \vec{A}_o}{\mathscr{r}}\, d\tau'$.

Practically speaking if we can solve the differential eq$^{\underline{n}}$ which assumes $\nabla \cdot \vec{A} = 0$ then we've imposed the Coulomb gauge. We worked through this before, once more:

$$\nabla \times \vec{B} = \mu_o \vec{J}$$

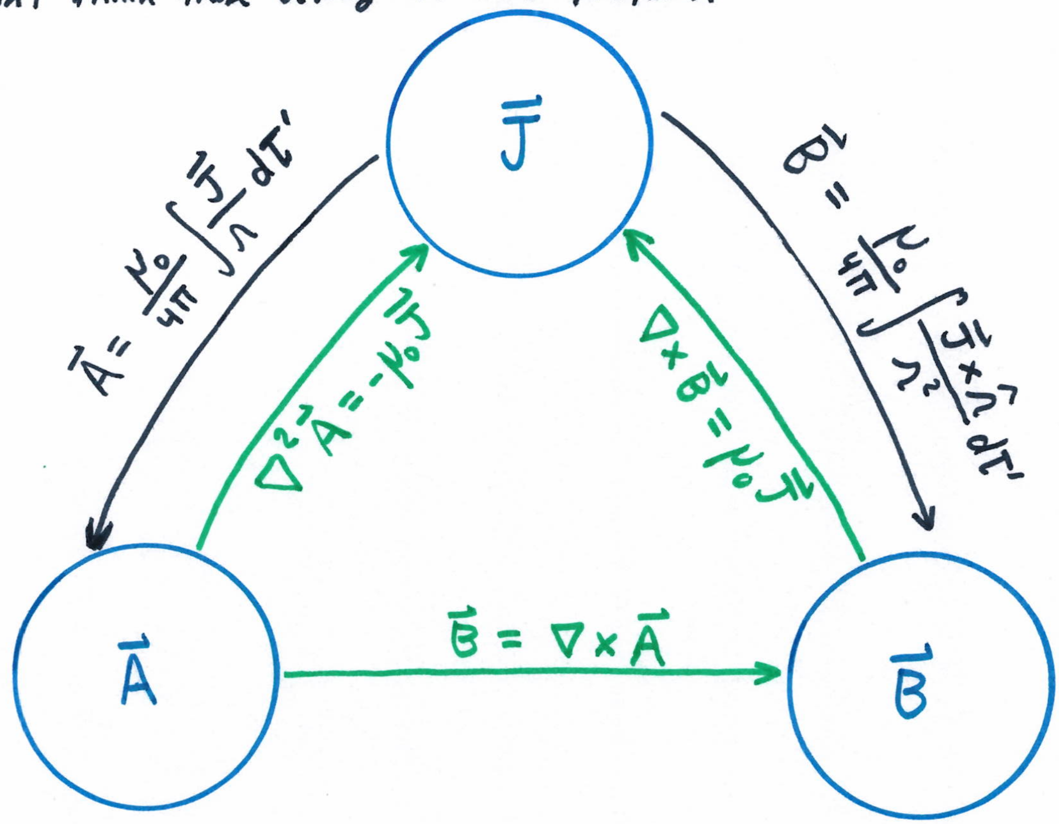
$$\nabla \times (\nabla \times \vec{A}) = \nabla(\cancel{\nabla \cdot \vec{A}})^{\,0} - \nabla^2 \vec{A} = \mu_o \vec{J}$$

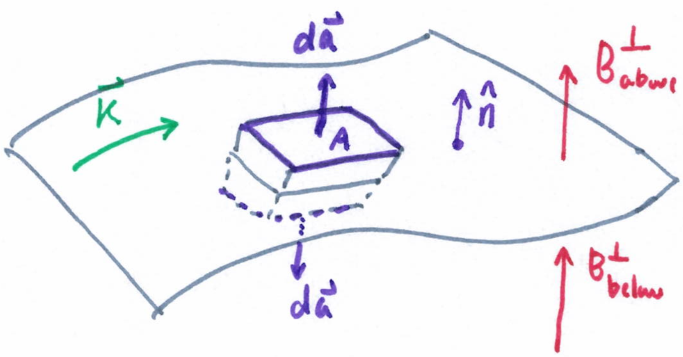
$$\nabla^2 \vec{A} = -\mu_o \vec{J} \longrightarrow \boxed{\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi}\int \frac{\vec{J}(\vec{r}\,')}{\mathscr{r}}\, d\tau'}$$

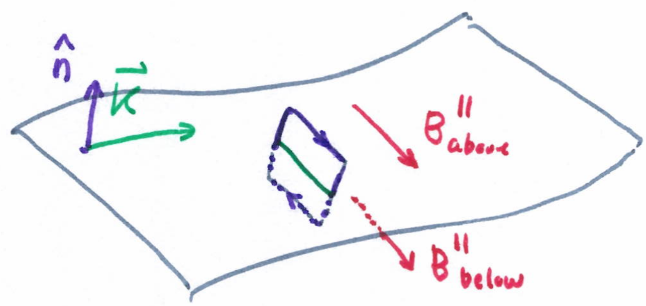
Remark: using the boxed formula merely implies $\nabla(\nabla \cdot \vec{A}) = 0$ so I suppose we might find $\nabla \cdot \vec{A}$ constant potentially...

BOUNDARY CONDITIONS:

GRIFFITH'S adds $\nabla \cdot \vec{B} = 0$ and $\nabla \cdot \vec{A} = 0$ to the diagram, but I don't think those belong in this picture.







$$\int \vec{B} \cdot d\vec{a} = 0$$

$$B^{\perp}_{above} - B^{\perp}_{below} = 0$$

$$\therefore \; \underline{B^{\perp}_{above} = B^{\perp}_{below}}.$$

$$\oint \vec{B} \cdot d\vec{l} = (B^{\parallel}_{above} - B^{\parallel}_{below})\, l = K l \mu_0$$

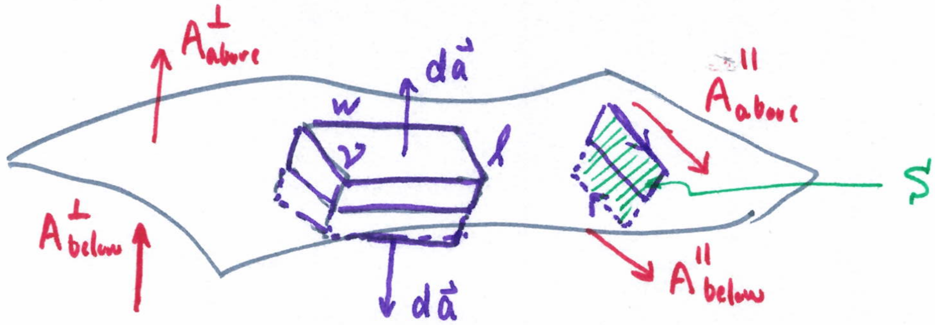
$$\underline{B^{\parallel}_{above} - B^{\parallel}_{below} = \mu_0 K}.$$

($\hat{K} \times \hat{n}$ points in $\parallel$-above direction)

$$\therefore \; \boxed{\vec{B}_{above} - \vec{B}_{below} = \mu_0 (\vec{K} \times \hat{n})}$$

BOUNDARY CONDITIONS FOR VECTOR POTENTIAL $\vec{A}$:

Assume $\nabla \cdot \vec{A} = 0$ then consider surface,



$$\int_{\partial \mathcal{V}} \vec{A} \cdot d\vec{a} = \int_{\mathcal{V}} (\nabla \cdot \vec{A})\, d\tau = 0$$

$$(A^{\perp}_{above} - A^{\perp}_{below})\, lw = 0 \Rightarrow \underline{A^{\perp}_{above} = A^{\perp}_{below}}$$

$$\int_{\partial S} \vec{A} \cdot d\vec{l} = \int_{S} (\nabla \times \vec{A}) \cdot d\vec{a} = \underbrace{\int_{S} \vec{B} \cdot d\vec{a}}_{} = 0$$

(let height of S shrink to zero to obtain BC)

"flux through an Amperian loop of vanishing thickness is zero" (p. 251 GRIFFITH's)

$$A^{\parallel}_{above}\, l - A^{\parallel}_{below}\, l = 0 \Rightarrow \underline{A^{\parallel}_{above} = A^{\parallel}_{below}}$$

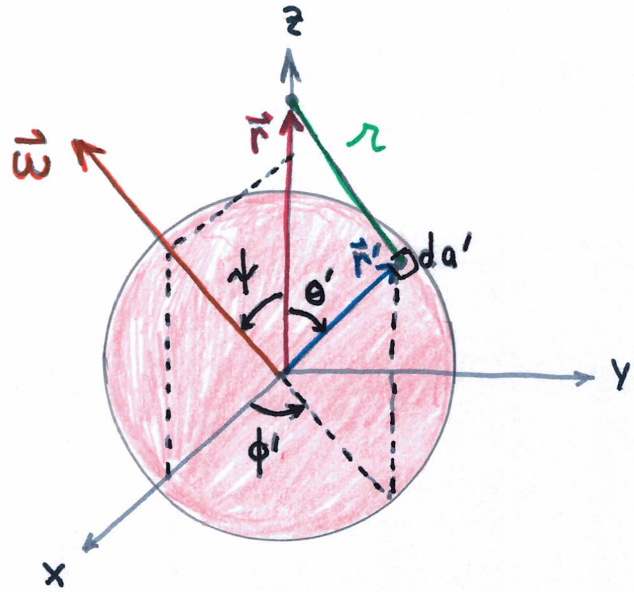
$$\therefore \quad \boxed{\vec{A}_{above} = \vec{A}_{below}} \leftarrow \text{continuity of } \vec{A}.$$

However, $\vec{B}_{above} - \vec{B}_{below} = \mu_0 (\vec{K} \times \hat{n}) = (\nabla \times \vec{A})_{above} - (\nabla \times \vec{A})_{below}$

$$\boxed{\frac{\partial \vec{A}_{above}}{\partial n} - \frac{\partial \vec{A}_{below}}{\partial n} = -\mu_0 \vec{K}}$$

could apply Stokes' Thm to Amperian loop like on previous page to derive.

E1 A spherical shell of radius R carries uniform surface charge σ and is set spinning at angular velocity $\vec{\omega}$. Find the vector potential due this current dist. at $\vec{r}$



- place field point on z-axis
- place $\vec{\omega}$ in xz-plane, at angle ψ from $\hat{z}$
- we wish to calculate

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{K}(\vec{r}')}{\mathscr{r}}\, da'$$

where $\vec{K} = \sigma \vec{V}$ and

$$da' = R^2 \sin\theta'\, d\theta'\, d\phi'$$

$$\mathscr{r} = \sqrt{R^2 + r^2 - 2Rr\cos(\theta')}$$

Recall the velocity for a point on a rigid body with ang. velocity $\vec{\omega}$ is,

$$\vec{V} = \vec{\omega} \times \vec{r}' = \langle \omega\sin\psi, 0, \omega\cos\psi \rangle \times \langle R\sin\theta'\cos\phi', R\sin\theta'\sin\phi', R\cos\theta' \rangle$$

$$\vec{V} = \langle \underbrace{-R\omega\cos\psi\sin\theta'\sin\phi'}_{*}, R\omega(\underbrace{\cos\psi\sin\theta'\cos\phi'}_{*} - \sin\psi\cos\theta'), \underbrace{R\omega\sin\psi\sin\theta'\sin\phi'}_{*} \rangle$$

All of $*$ integrates to zero since $\int_0^{2\pi}\sin\phi'\, d\phi' = \int_0^{2\pi}\cos\phi'\, d\phi' = 0$

Thus,

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\sigma\langle 0, -R\omega\sin\psi\cos\theta', 0\rangle R^2\sin\theta'\, d\theta'\, d\phi'}{\sqrt{R^2 + r^2 - 2rR\cos\theta'}}$$

$$= \frac{-\mu_0 R^3 \sigma \omega \sin\psi}{2}\left(\int_0^{\pi} \frac{\sin\theta'\cos\theta'\, d\theta'}{\sqrt{R^2 + r^2 - 2Rr\cos\theta'}}\right)\hat{y}$$

E1 continued

$$\int_0^{\pi} \frac{\sin\theta\cos\theta\, d\theta}{\sqrt{R^2+r^2-2Rr\cos\theta}} = \int_1^{-1} \frac{u\,(-du)}{\sqrt{R^2+r^2-2ru}}$$

$$\boxed{\begin{array}{l} u = \cos\theta \\ u(0) = 1,\ u(\pi) = -1 \\ du = -\sin\theta\, d\theta \end{array}}$$

$$= \int_{-1}^{1} \frac{u\,du}{\sqrt{R^2+r^2-2ruR}}$$

$$\int \frac{u\,du}{\sqrt{\underbrace{R^2+r^2-2rRu}}} = \int \frac{1}{\sqrt{w}}\left(\frac{R^2+r^2-w}{2rR}\right)\left(\frac{-dw}{2rR}\right)$$

$$w = R^2+r^2-2rRu$$

$$u = \frac{R^2+r^2-w}{2rR}$$

$$du = \frac{-dw}{2rR}$$

$$= \frac{1}{4r^2R^2}\int \left(\sqrt{w} - (R^2+r^2)\,w^{-1/2}\right) dw$$

$$= \frac{1}{4r^2R^2}\left[\frac{2}{3}w^{3/2} - 2(R^2+r^2)\,w^{1/2}\right]$$

$$= \frac{1}{2r^2R^2}\left[\frac{1}{3}w - (R^2+r^2)\right] w^{1/2}$$

$$= \frac{1}{2r^2R^2}\left[\frac{R^2+r^2-2rRu-3R^2-3r^2}{3}\right] w^{1/2}$$

$$= \frac{1}{2r^2R^2}\left[\frac{-2R^2-2r^2-2rRu}{3}\right] w^{1/2}$$

$$= \frac{-1}{3r^2R^2}\left(R^2+r^2+rRu\right)\sqrt{R^2+r^2-2rRu}$$

E1 continued

$$\int_{-1}^{1}\frac{u\,du}{\sqrt{R^2+r^2-2rRu}} = \frac{-1}{3r^2R^2}\left(R^2+r^2+rRu\right)\sqrt{R^2+r^2-2rRu}\,\Bigg|_{-1}^{1}$$

$$= \frac{-1}{3r^2R^2}\left(R^2+r^2+rR\right)\sqrt{R^2+r^2-2rR} + \frac{1}{3r^2R^2}\left(R^2+r^2-rR\right)\sqrt{R^2+r^2+2rR}$$

$$= \frac{-1}{3R^2r^2}\left[\left(R^2+r^2+rR\right)\sqrt{(r-R)^2} - \left(R^2+r^2-rR\right)\sqrt{(r+R)^2}\right]$$

$$= \frac{-1}{3R^2r^2}\left[\left(R^2+r^2+rR\right)|r-R| - \left(R^2+r^2-rR\right)(r+R)\right]$$

$$= \begin{cases} \dfrac{-1}{3R^2r^2}\left[\left(R^2+r^2+rR\right)(r-R) - \left(R^2+r^2-rR\right)(r+R)\right] & : r \geq R \\ \dfrac{-1}{3R^2r^2}\left[\left(R^2+r^2+rR\right)(R-r) - \left(R^2+r^2-rR\right)(r+R)\right] & : r \leq R \end{cases}$$

$$= \begin{cases} \dfrac{-1}{3R^2r^2}\left[R^2r + r^3 + r^2R - R^3 - r^2R - rR^2 - \left(R^2r + r^3 - r^2R + R^3 + r^2R - rR^2\right)\right] & : r \geq R \\ \dfrac{-1}{3R^2r^2}\left[R^3 + r^2R + rR^2 - rR^2 - r^3 - r^2R - \left(R^2r + r^3 - r^2R + R^3 + r^2R - rR^2\right)\right] & : r \leq R \end{cases}$$

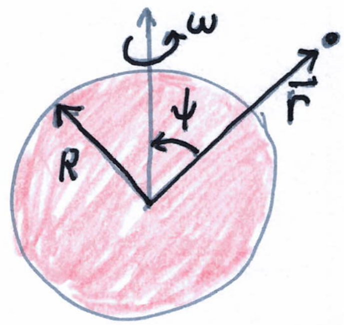
$$= \begin{cases} \dfrac{2R^3}{3R^2r^2} & : r \geq R \\ \dfrac{2r^3}{3R^2r^2} & : r \leq R \end{cases}$$

$$= \begin{cases} \dfrac{2R}{3r^2} & : r \geq R \\ \dfrac{2r}{3R^2} & : r \leq R \end{cases}$$

$$\vec{\omega}\times\vec{r} = \left(-\omega r \sin\psi\right)\hat{y}$$

$$\vec{A}(\vec{r}) = \frac{-\mu_0 R^3 \sigma \omega \sin\psi}{2}\left(\int_{-1}^{1}\frac{u\,du}{\sqrt{R^2+r^2-2Rru}}\right)\hat{y} = \begin{cases} \dfrac{\mu_0 R \sigma}{3}\left(\vec{\omega}\times\vec{r}\right) & : r \leq R \\ \dfrac{\mu_0 R^4 \sigma}{3r^3}\left(\vec{\omega}\times\vec{r}\right) & : r \geq R \end{cases}$$

Now we can reformulate our result with $\vec{\omega}$ as pictured



$$\vec{\omega}\times\vec{r} = \omega r\,\hat{z}\times\hat{r} = \omega r\sin\theta\,\hat{\phi}$$

$$\vec{A}(\vec{r}) = \begin{cases} \dfrac{\mu_0 R\sigma}{3}(\vec{\omega}\times\vec{r}) & : r\le R \\ \dfrac{\mu_0 R^4\sigma}{3r^3}(\vec{\omega}\times\vec{r}) & : r\ge R \end{cases}$$

$$\equiv \begin{cases} \dfrac{\mu_0 R\omega\sigma}{3} r\sin\theta\,\hat{\phi} & : r\le R \\ \dfrac{\mu_0 R^4\omega\sigma}{3r^2}\sin\theta\,\hat{\phi} & : r\ge R \end{cases}$$

For $r\le R$,

$$\vec{B} = \nabla\times\vec{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi)\right]\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\hat{\theta}$$

$$= \frac{\mu_0 R\omega\sigma}{3\sin\theta}\frac{\partial}{\partial\theta}\left[\sin^2\theta\right]\hat{r} - \frac{\mu_0 R\omega\sigma}{3r}\frac{\partial}{\partial r}\left[r^2\sin\theta\right]\hat{\theta}$$

$$= \frac{\mu_0 R\omega\sigma}{3\sin\theta}(2\sin\theta\cos\theta)\hat{r} - \frac{2\mu_0 R\omega\sigma r\sin\theta}{3r}\hat{\theta}$$

$$= \frac{2\mu_0 R\omega\sigma}{3}(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})$$

$$= \frac{2\mu_0 R\omega\sigma}{3}\hat{z}$$

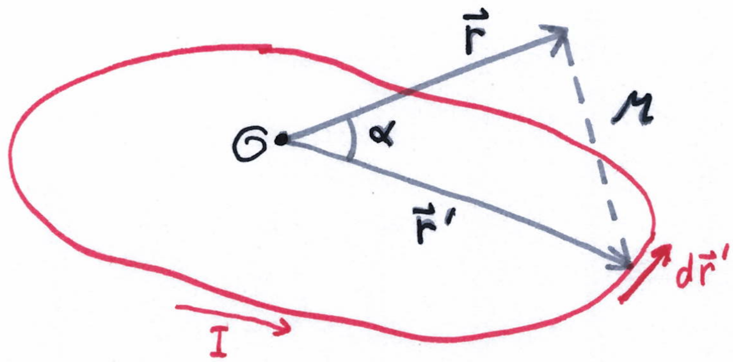
$$= \frac{2}{3}\mu_0\sigma R\vec{\omega}$$

$\vec{B}$ field uniform within the spinning charged spherical shell.

MULTIPOLE & DIPOLE FOR MAGNETOSTATICS:

The crucial mathematical observation remains the same as was seen for electrostatics:

$$\boxed{\frac{1}{\eta} = \frac{1}{\sqrt{r^2+(r')^2-2rr'\cos\alpha}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)}$$



Let's apply the Legendre polynomial expansion to the vector potential of a current loop,

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\oint \frac{1}{\eta}\, d\vec{l}\,'$$

$$= \frac{\mu_0 I}{4\pi}\oint \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)\, d\vec{l}$$

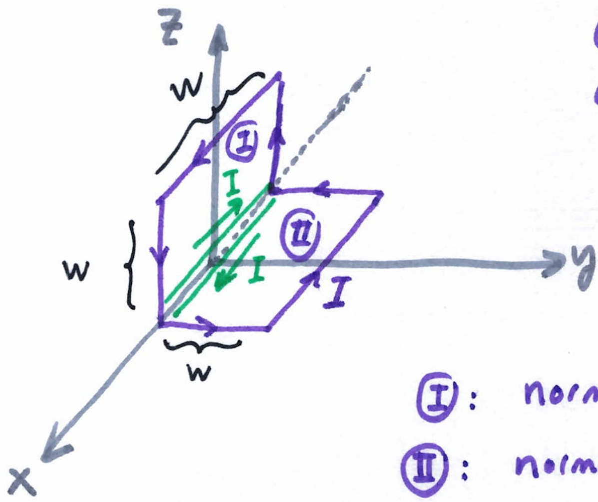
$$= \frac{\mu_0 I}{4\pi}\left[\underbrace{\frac{1}{r}\oint d\vec{l}\,'}_{\substack{\text{monopole}\\ \text{term integrates}\\ \text{to zero}}} + \underbrace{\frac{1}{r^2}\oint r'\cos\alpha\, d\vec{l}\,'}_{\text{dipole}} + \underbrace{\frac{1}{r^3}\oint (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)d\vec{l}\,'}_{\text{quadrapole}} + \dots\right]$$

$$\vec{A}_{dipole}(\vec{r}) = \frac{\mu_0 I}{4\pi r^2}\oint r'\cos\alpha\, d\vec{l}\,' = \frac{\mu_0 I}{4\pi r^2}\oint (\hat{r}\cdot\vec{r}\,')\, d\vec{l}\,'$$

$$= \frac{\mu_0 I}{4\pi r^2}\left(-\hat{r}\times\int d\vec{a}\,'\right)$$

$$\boxed{\underbrace{\vec{m} = I\int d\vec{a}\,' = I\vec{a}}_{\text{(dipole moment of current loop)}} \quad \& \quad \vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}}$$

E2 Something a bit easier,



Given current loop pictured, we can add two currents to see this as two loops

I: normal $\hat{y}$ and $\vec{A}_I = w^2 \hat{y}$

II: normal $\hat{z}$ and $\vec{A}_{II} = w^2 \hat{z}$

$$\vec{m} = I\vec{A}_I + I\vec{A}_{II} = Iw^2\hat{y} + Iw^2\hat{z} = \boxed{Iw^2(\hat{y}+\hat{z}) = \vec{m}}$$

Pure vs. Physical Dipole Magnetic Field

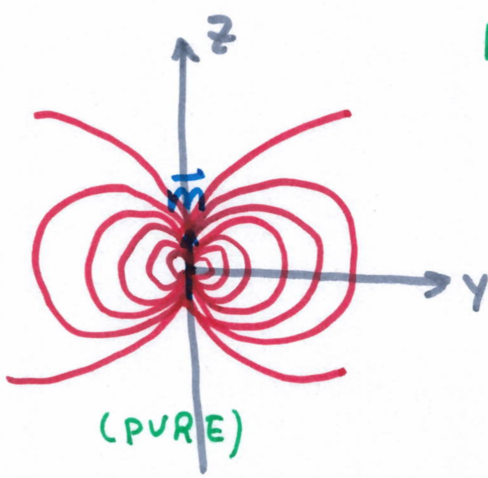
Placing $\vec{m}$ at origin and $\vec{m} = m\hat{z}$ then

$$\boxed{\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\left(\frac{m\sin\theta}{r^2}\right)\hat{\phi}}$$

From which we calculate, (same formula as $\vec{E}_{dip}$)

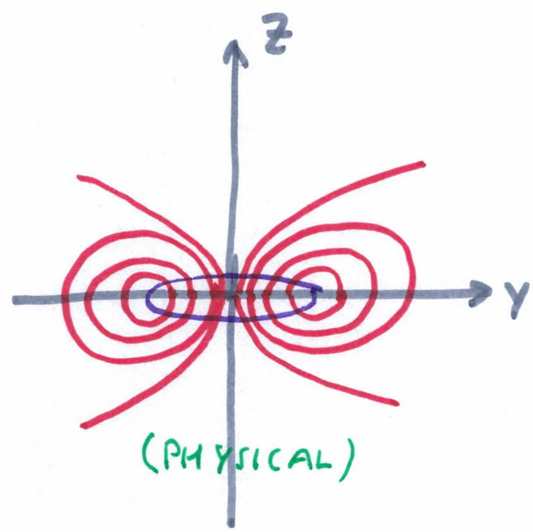
$$\boxed{\vec{B}_{dip}(\vec{r}) = \nabla\times\vec{A} = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)}$$

$$\hookrightarrow \quad \vec{B}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(m\cdot\hat{r})\hat{r} - \vec{m}\right]$$



(PURE)

FAR AWAY LOOK SAME.



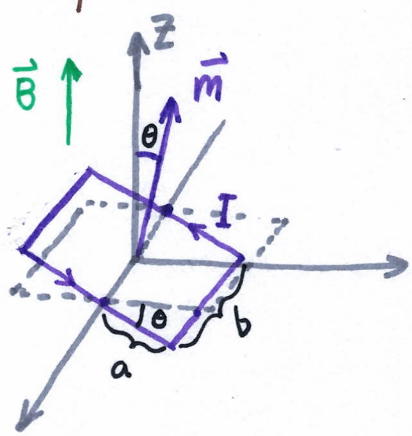
(PHYSICAL)

Magnetic Fields in Matter:

What is a magnet? When a material is magnetized it has a bulk magnetic dipole moment, or thinking locally, a dipole moment per unit-volume. There are three common cases:

1. magnetization // to $\vec{B}$ is a <u>paramagnet</u>.
2. magnetization opposite $\vec{B}$ is a <u>diamagnet</u>.
3. magnetization retained after $\vec{B}$ removed (like IRON) are <u>ferromagnets</u>

$\boxed{E3}$ torque on current loop $\vec{\tau} = \vec{m} \times \vec{B}$



- net force is zero, but

$$\vec{\tau} = (a F \sin\theta)\hat{x}$$

$$F = IbB$$

$$\vec{\tau} = IabB\sin\theta = (mB\sin\theta)\hat{x}$$

$$\therefore \ \vec{\tau} = \vec{m} \times \vec{B}$$

<u>Remark</u>: if we consider the electron orbiting in an atom then an applied $\vec{B}$-field will tend to twist the atoms to align a certain way (more next Lecture)

MAGNETIZATION

$\boxed{\text{Def}^{n}/\ \vec{M} \text{ is magnetic dipole moment per unit volume}}$

The vector potential for single dipole $\vec{A} = \frac{\mu_0}{4\pi}\frac{m \times \hat{\imath}}{\imath^2}$ thus for a collection,

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{M}(\vec{r}\,') \times \hat{\imath}}{\imath^2}\, d\tau'$$

Then after a short calculation,

$$\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{\vec{J}_b(\vec{r}\,')}{\imath}\, d\tau' + \frac{\mu_0}{4\pi}\oint_{\partial\mathcal{V}} \frac{\vec{K}_b(\vec{r}\,')}{\imath}\, da'}$$

Def^{n}/ $\vec{J}_b = \nabla \times \vec{M}$ (bound volume current)

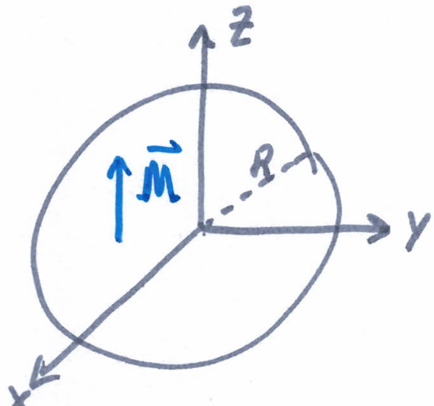
$\vec{K}_b = \vec{M} \times \hat{n}$ (bound surface current)

$\boxed{\text{E4}}$ Find magnetic field of uniformly magnetized sphere

Let $\vec{M} = M\hat{z}$ then $\vec{J}_b = \nabla \times \vec{M} = 0$ whereas

$$\vec{K}_b = \vec{M} \times \hat{r} = M \sin\theta\, \hat{\phi}$$

$$\vec{K} = \sigma \vec{v} = \sigma \omega R \sin\theta\, \hat{\phi}$$



Identify $M = \sigma\omega R$ to obtain

$$\boxed{\vec{B} = \frac{2}{3}\mu_0 \vec{M}}$$ (internal field is uniform)

then outside, it is as if $\vec{m} = \left(\frac{4}{3}\pi R^3\right)\vec{M}$ at origin.