Lecture 20: General Relativity: Tetrad Formalism ala Cartan

Typically we've used $T_pM = \text{span}\{\frac{\partial}{\partial x^i}\big|_p\}$ or $\frac{\partial}{\partial x^i} : C^\infty(U) \longrightarrow C^\infty(U)$ to describe vectors at a point $p \in M$ or vector fields on $U \subseteq M$ where $x^i : U \longrightarrow \mathbb{R}^n$ is a coordinate chart on U. In G.R. we have $n=4$ and we use x^μ to indicate $\mu = 0,1,2,3$ subject our usual conventions. In short, $\partial_\mu = \frac{\partial}{\partial x^\mu}$ and dx^μ give bases for tangent and cotangent spaces to our spacetime manifold. For a general coord. system the metric tensor's components $g_{\mu\nu} = g(\partial_\mu, \partial_\nu)$ need not be anything too nice (well g is a metric, so $\det(g_{\mu\nu}) \neq 0$ and $g_{\mu\nu} = g_{\nu\mu}$ etc... so they're are some conditions!)

Def^n/ vector fields e_a are called a <u>tetrad</u> if $a = 0,1,2,3$ satisfy

$$g(e_a, e_b) = \eta_{ab} \quad \text{where} \quad \eta_{ab} = -\delta_{a,0}\delta_{b,0} + \delta_{a,1}\delta_{b,1} + \delta_{a,2}\delta_{b,2} + \delta_{a,3}\delta_{b,3}$$

throughout some $U \subseteq M$. We also say $\{e_a\}$ forms a <u>vielbein</u> on U.

$$(\eta_{ab}) = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

<u>Remark</u>: it is not generally possible to find a ~~tetrad~~ vielbein which covers all of a manifold.

<u>Vielbein</u>: many legs

For instance the sphere does not permit the assignment of an everywhere nonzero smooth vect. field. The tetrad is a local construction.

<u>Remark</u>: Carroll uses $\hat{e}_{(a)}$ whereas I'll use e_a and add parentheses if in doubt about interpretation for formula...

Def^{n}

$e_\mu = \partial_\mu = \frac{\partial}{\partial x^\mu}$ (coordinate derivations)

e_a (satisfy $g(e_a, e_b) = \eta_{ab}$, might be ~~noncoord. derivs~~ impossible to write e_a as coord. derivatives on their domain.

Def^{n}/ $e_\mu = e_\mu{}^a e_a$ and $e_a = e^\mu{}_a e_\mu$

The $(n\times n)$-invertible matrix $(e_\mu{}^a)$ is also called the tetrad since it is the coordinates of e_μ in the e_a-basis. Likewise, $(e^\mu{}_a)$ ~~are~~ contain the coordinates of e_a in the e_μ-basis.

Observe $e_\mu = e_\mu{}^a e_a = e_\mu{}^a e^\nu{}_a e_\nu \Rightarrow \underline{e_\mu{}^a e^\nu{}_a = \delta_\mu{}^\nu}$ (1)

Likewise, $e_a = e^\mu{}_a e_\mu = e^\mu{}_a e_\mu{}^b e_b \Rightarrow \underline{e^\mu{}_a e_\mu{}^b = \delta_a{}^b}$ (2)

Furthermore, $g(e_a, e_b) = \eta_{ab}$

$g(e^\mu{}_a e_\mu, e^\nu{}_b e_\nu) = \eta_{ab}$

$e^\mu{}_a e^\nu{}_b\, g(e_\mu, e_\nu) = \eta_{ab} \Rightarrow \underline{e^\mu{}_a e^\nu{}_b\, g_{\mu\nu} = \eta_{ab}}$ (3)

Finally, $\underbrace{e_\alpha{}^a e_\beta{}^b e^\mu{}_a e^\nu{}_b}_{\delta_\alpha{}^\mu\ \delta_\beta{}^\nu} g_{\mu\nu} = e_\alpha{}^a e_\beta{}^b \eta_{ab} \Rightarrow \underline{g_{\alpha\beta} = e_\alpha{}^a e_\beta{}^b \eta_{ab}}$ (4)

Th$^{\underline{m}}$/ If e_a is tetrad frame with $g(e_a, e_b) = \eta_{ab}$ and we define coordinate change matrices $e_\mu{}^a$ and $e^\mu{}_a$ to convert from tetrad e_a to coord. frame $e_\mu = \frac{\partial}{\partial x^\mu}$ ~~then~~ meaing

$$e_\mu = e_\mu{}^a e_a \quad \text{and} \quad e_a = e^\mu{}_a e_\mu \quad \text{then,}$$

1. $e_\mu{}^a e^\nu{}_a = \delta^\nu_\mu$
2. $e^\mu{}_a e_\mu{}^b = \delta^b_a$
3. $\eta_{ab} = e^\mu{}_a e^\nu{}_b g_{\mu\nu}$
4. $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}$

Remark: a, b are "Lorentz indices" whereas μ, ν are general indices. ← or better yet "curved" indices

The condition $e_\mu = \frac{\partial}{\partial x^\mu}$ was not used to derive ①, ②, ③, ④ so this is more genenercal than advertised.

Def$^{\underline{n}}$/ compatible basis of one-forms to given e_μ can be denoted θ^μ where $\theta^\mu(e_\nu) = \delta^\mu_\nu$ serves to define θ^μ as a linear function of vectors (at a point, or on a subset in which case we'd call θ^μ a one-form-field) Likewise, $\theta^a(e_b) = \delta^a_b$. Alternatively, $\theta^\mu = dx^\mu$, $e_\mu = \partial_\mu$ and $dx^\mu(\partial_\nu) = \delta^\mu_\nu$

Remark: the coordinate derivations $\frac{\partial}{\partial x^\mu}$ and differentials dx^μ give example of frame and compatible coframe; $dx^\mu\left(\frac{\partial}{\partial x^\nu}\right) = \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu$.

Coordinate Change for the Coframe

$$\Theta^a(e_b) = \delta^a_b \;\Rightarrow\; \Theta^a({e^\mu}_b\, e_\mu) = \delta^a_b$$

$$\Rightarrow\; {e^\mu}_b\, \Theta^a(e_\mu) = \delta^a_b$$

$$\Rightarrow\; \underbrace{\boxed{{e_\nu}^b\, {e^\mu}_b}}_{\delta_\nu^\mu}\, \Theta^a(e_\mu) = \delta^a_b\, {e_\nu}^b$$

$$\Rightarrow\; \boxed{\Theta^a(e_\nu) = {e_\nu}^a}$$

Likewise,

$$\Theta^\mu(e_\nu) = \delta^\mu_\nu \;\Rightarrow\; \Theta^\mu({e_\nu}^a\, e_a) = \delta^\mu_\nu$$

$$\Rightarrow\; \underbrace{\boxed{{e^\nu}_b\, {e_\nu}^a}}_{\delta_b^a}\, \Theta^\mu(e_a) = {e^\nu}_b\, \delta^\mu_\nu$$

$$\Rightarrow\; \boxed{\Theta^\mu(e_b) = {e^\mu}_b}$$

$$\boxed{\text{Th}^{\underline{m}}\text{/ with notation as given above, } \Theta^\mu = {e^\mu}_a\, \Theta^a \text{ and } \Theta^a = {e_\mu}^a\, \Theta^\mu}$$

<u>Proof</u>: $\Theta^\mu = \Theta^\mu(e_a)\, e^a = {e^\mu}_a\, e^a$

$\Theta^a = \Theta^a(e_\mu)\, e^\mu = {e_\mu}^a\, e^\mu$. //

(here I'm using $\alpha : V \rightarrow \mathbb{R}$ has $\alpha = \sum c_i e_i$ then

$\alpha(e^j) = \sum c_i\, e_i(e^j) = \sum c_i\, \delta_i^j = c_j \;\therefore\; c_j = \alpha(e^j)$

hence $\alpha = \sum_i \alpha(e_i)\, e^i$ is an identity linking frame/coframe basis expansions)

FLAT AND CURVED INDEX JUGGLING

A vector V can be written $V = V^\mu e_\mu$ or $V = V^a e_a$ then the components of V are related by the vielbein

$$V = V^\mu e_\mu = V^\mu e_\mu{}^a e_a = V^a e_a \Rightarrow \boxed{V^a = e_\mu{}^a V^\mu}$$

$$V = V^a e_a = V^a e^\mu{}_a e_\mu = V^\mu e_\mu \Rightarrow \boxed{V^\mu = e^\mu{}_a V^a}$$

Higher tensors work similarly,

$$\begin{aligned} T^a{}_b = T(e^a, e_b) &= T(e_\mu{}^a e^\mu, e^\nu{}_b e_\nu) \\ &= e_\mu{}^a e^\nu{}_b T(e^\mu, e_\nu) \\ &= e_\mu{}^a e^\nu{}_b T^\mu{}_\nu \end{aligned}$$

Following the pattern, (consistent with what we derived earlier) (② or Th^{m} on ③)

$$g_{\mu\nu} = e_\mu{}^a e_\nu{}^b g_{ab} \quad \text{or} \quad g_{ab} = e^\mu{}_a e^\nu{}_b g_{\mu\nu}$$

{ Th^{m}/ Thinking of tensor $e = e_\nu{}^a dx^\nu \otimes e_a$ we can see vierbeins as components of this tensor. (weird to use diff. coord. in this fashion...) }

$$\boxed{e^\mu{}_a = g^{\mu\nu} \eta_{ab} e_\nu{}^b}$$

$\theta^\mu(e_a) = e^\mu{}_a$ from ④

$\theta^b(e_\nu) = e_\nu{}^b$ from ④

$$e^{\mu}{}_{a} = \Theta^{\mu}(e_a) = e^{\mu}{}_{b}\,\Theta^{b}(e^{\nu}{}_{a}\,e_{\nu})$$
$$= e^{\mu}{}_{b}\,e^{\nu}{}_{a}\,\Theta^{b}(e_{\nu})$$
$$= e^{\mu}{}_{b}\,\underbrace{e^{\nu}{}_{a}\,e_{\nu}{}^{b}}_{\delta^{b}_{a}} = e^{\mu}{}_{b}\,\delta^{b}_{a} = e^{\mu}{}_{a}\,.$$

YEP.

$$g^{\mu\nu}\,g_{\nu\sigma} = \delta^{\mu}_{\sigma} \quad\Rightarrow\quad g^{\mu\nu}(e_{\nu}{}^{a}\,e_{\sigma}{}^{b}\,\eta_{ab}) = \delta^{\mu}_{\sigma}$$
$$\Rightarrow\quad \underbrace{e^{\sigma}{}_{c}\,g^{\mu\nu}\,e_{\nu}{}^{a}\,e_{\sigma}{}^{b}}_{\delta^{b}_{c}}\,\eta_{ab} = e^{\sigma}{}_{c}\,\delta^{\mu}_{\sigma} = e^{\mu}{}_{c}$$
$$\Rightarrow$$
$$\Rightarrow\quad g^{\mu\nu}\,\eta_{ac}\,e_{\nu}{}^{a} = e^{\mu}{}_{c}$$
$$\Rightarrow\quad g^{\mu\nu}\,\eta_{ab}\,e_{\nu}{}^{a} = e^{\mu}{}_{b}$$
$$\Rightarrow\quad \boxed{g^{\mu\nu}\,\eta_{ab}\,e_{\nu}{}^{b} = e^{\mu}{}_{a}}$$

LOCAL LORENTZ TRANSFORMATIONS VS. GENERAL COORD. TRANSFORMATIONS

Given a tetrad over some region of spacetime we can form a host of other tetrads by performing LLT's

Def^n/ e_a and $e_{a'}$ are related by LLT if $g(e_a, e_b) = \eta_{ab}$ and $g(e_{a'}, e_{b'}) = \eta_{a'b'}$ where both η_{ab} and $\eta_{a'b'}$ denote Minkowski metric.

Notation: $e_{a'} = \Lambda^a{}_{a'} e_a$ or to be more explicit, $\underbrace{e_{a'}(x) = \Lambda^a{}_{a'}(x) e_a(x)}$

(matrix $\Lambda^a{}_{a'}$ can vary with x, ok it's a matrix of functions)

The terminology Lorentz is justified,

$$\begin{aligned}\eta_{a'b'} &= g(e_{a'}, e_{b'}) \\ &= g(\Lambda^a{}_{a'} e_a, \Lambda^b{}_{b'} e_b) \\ &= \Lambda^a{}_{a'} \Lambda^b{}_{b'} g(e_a, e_b) \\ &= \Lambda^a{}_{a'} \Lambda^b{}_{b'} \eta_{ab}\end{aligned}$$

Notation: $e_a = \Lambda^{a'}{}_a e_{a'}$ where we have $\Lambda^a{}_{b'} \Lambda^{b'}{}_b = \delta^a_b$ and also $\Lambda^{a'}{}_b \Lambda^b{}_{b'} = \delta^{a'}_{b'}$

General Coord. Transformations (GCTs) need not preserve form of metric, Carroll mentions we can use mixture of flat and curved indices,

$$T^{a'\mu'}{}_{b'\nu'} = \Lambda^{a'}{}_a \frac{\partial x^{\mu'}}{\partial x^\mu} \Lambda^b{}_{b'} \frac{\partial x^\nu}{\partial x^{\nu'}} T^{a\mu}{}_{b\nu}$$

Ok, now to the eq^n's & constructions which form GR, but <u>differentiation</u> well, covariant diff. requires some care ↗

SPIN CONNECTION: (for Lorentz indices)

$$\boxed{\text{Def}^{n}/\quad \nabla_\mu X^a{}_b = \partial_\mu X^a{}_b + \omega_\mu{}^a{}_c X^c{}_b - \omega_\mu{}^c{}_b X^a{}_c}$$

How do the spin connection coefficients $\omega_\mu{}^a{}_b$ relate to $\Gamma^\nu_{\mu\lambda}$ coefficients?
We calculate as does Carroll on p. 486 in Appendix J; for vector X,

$$\nabla X = (\nabla_\mu X^\nu)\, dx^\mu \otimes \partial_\nu = (\underbrace{\partial_\mu X^\nu + \Gamma^\nu_{\mu\lambda} X^\lambda}_{*})\, dx^\mu \otimes \partial_\nu$$

In mixed basis,

$$\nabla X = (\nabla_\mu X^a)\, dx^\mu \otimes e_a = (\partial_\mu X^a + \omega_\mu{}^a{}_b X^b)\, dx^\mu \otimes e_a$$

$$= \left(\partial_\mu (e_\nu{}^a X^\nu) + \omega_\mu{}^a{}_b\, e_\nu{}^b X^\nu\right) dx^\mu \otimes (e^\lambda{}_a \partial_\lambda)$$

$$= \Big[(\partial_\mu X^\nu)\underbrace{e_\nu{}^a e^\lambda{}_a}_{\delta^\lambda_\nu} + \partial_\mu(e_\nu{}^a) X^\nu e^\lambda{}_a + \omega_\mu{}^a{}_b\, e_\nu{}^b e^\lambda{}_a X^\nu\Big] dx^\mu \otimes \partial_\lambda$$

$$= \Big[\partial_\mu X^\lambda + X^\nu \partial_\mu (e_\nu{}^a)\, e^\lambda{}_a + \omega_\mu{}^a{}_b\, e_\nu{}^b e^\lambda{}_a X^\nu\Big] dx^\mu \otimes \partial_\lambda$$

$$= \Big[\partial_\mu X^\nu + \big(\underbrace{\partial_\mu (e_\lambda{}^a)\, e^\nu{}_a + \omega_\mu{}^a{}_b\, e_\lambda{}^b e^\nu{}_a}\big) X^\lambda\Big] dx^\mu \otimes \partial_\nu$$

comparing with $*$ we find $\longrightarrow$ $\Gamma^\nu_{\mu\lambda} = e^\nu{}_a \partial_\mu (e_\lambda{}^a) + \omega_\mu{}^a{}_b\, e_\lambda{}^b e^\nu{}_a$

We found $\Gamma^{\nu}_{\mu\lambda} = e^{\nu}{}_{a}\,\partial_\mu (e_\lambda{}^{a}) + \omega_\mu{}^{a}{}_{b}\, e_\lambda{}^{b} e^{\nu}{}_{a}$

$$e_\lambda{}^{b} e^{\nu}{}_{a}\, \omega_\mu{}^{a}{}_{b} = \Gamma^{\nu}_{\mu\lambda} - e^{\nu}_{a}\,\partial_\mu (e_\lambda{}^{a})$$

Multiply by inverse tetrads,

$$(e^{\lambda}{}_{a'}\, e_\nu{}^{b'})(e_\lambda{}^{b} e^{\nu}{}_{a}\, \omega_\mu{}^{a}{}_{b}) = (e^{\lambda}{}_{a'}\, e_\nu{}^{b'})(\Gamma^{\nu}_{\mu\lambda} - \underbrace{e^{\nu}{}_{a}}_{}\,\partial_\mu (e_\lambda{}^{a}))$$

with $\delta_a^{b'}$ above the pairing of $e_\nu{}^{b'}$ and $e^{\nu}{}_a$.

$$\delta^{b}_{a'}\,\delta_a^{b'}\,\omega_\mu{}^{a}{}_{b} = \omega_\mu{}^{b'}{}_{a'} = e^{\lambda}{}_{a'}\, e_\nu{}^{b'}\,\Gamma^{\nu}_{\mu\lambda} - e^{\lambda}{}_{a'}\,\partial_\mu (e_\lambda{}^{b'})$$

$$\therefore \boxed{\omega_\mu{}^{a}{}_{b} = e_\nu{}^{a} e^{\lambda}{}_{b}\,\Gamma^{\nu}_{\mu\lambda} - e^{\lambda}{}_{b}\,\partial_\mu (e_\lambda{}^{a})} \quad *$$

$\boxed{\text{Thm}/\ \nabla_\mu e_\nu{}^{a} = 0}$ (Carroll mentions this is sometimes known as the tetrad postulate)

Proof: $\nabla_\mu e_\nu{}^{a} = \partial_\mu e_\nu{}^{a} - \Gamma^{\lambda}_{\mu\nu} e_\lambda{}^{a} + \omega_\mu{}^{a}{}_{b'} e_\nu{}^{b'}$

Then $\underline{e^{\lambda}{}_{b}\,\partial_\mu (e_\lambda{}^{a})} = -\omega_\mu{}^{a}{}_{b} + e_\nu{}^{a} e^{\lambda}{}_{b}\,\Gamma^{\nu}_{\mu\lambda}$ from *

multiply by $e^{\nu}{}_{b}$ summing over ν

$$e^{\nu}{}_{b}\,\nabla_\mu e_\nu{}^{a} = e^{\nu}{}_{b}\,\partial_\mu e_\nu{}^{a} - \Gamma^{\lambda}_{\mu\nu} e^{\nu}{}_{b} e_\lambda{}^{a} + \underline{e^{\nu}{}_{b}}\,\omega_\mu{}^{a}{}_{b'}\underline{e_\nu{}^{b'}}$$

$$= \underline{e^{\nu}{}_{b}\,\partial_\mu e_\nu{}^{a}} - \Gamma^{\lambda}_{\mu\nu} e^{\nu}{}_{b} e_\lambda{}^{a} + \omega_\mu{}^{a}{}_{b}$$

$$= -\omega_\mu{}^{a}{}_{b} + e_\nu{}^{a} e^{\lambda}{}_{b}\,\Gamma^{\nu}_{\mu\lambda} - \Gamma^{\lambda}_{\mu\nu} e^{\nu}{}_{b} e_\lambda{}^{a} + \omega_\mu{}^{a}{}_{b}$$

$$= 0 \Rightarrow \nabla_\mu e_\nu{}^{a} = 0. \ //$$

Thm/ $\omega_\mu{}^{a'}{}_{b'} = \Lambda^{a'}{}_a \Lambda^b{}_{b'} \omega_\mu{}^a{}_b - \Lambda^c{}_{b'} \partial_\mu \Lambda^{a'}{}_c$ — Spin connection transforms inhomogeneously.

Proof: possibly begin with $*$ on pg. 8 and use the transformation formulae we've found. //

QUESTION: why the tetrad formalism?
- allows covariant derivatives of spinors
- changes our viewpoint for tensors, we can think of tensors as tensor-valued differential forms
  - $\Sigma_\mu{}^a$ is vector-valued one-form
  - $A_{\mu\nu}{}^a{}_b$ antisymmetric in $\mu\nu$ gives (1,1)-tensor valued two-form

Exterior Derivatives of Tensor-Valued Differential Forms, naively,

$$(d\Sigma)_{\mu\nu}{}^a = \partial_\mu \Sigma_\nu{}^a - \partial_\nu \Sigma_\mu{}^a$$

transforms as 2-form for GCT, but fails to transform as vector under LLT

CLAIM/ $(d\Sigma)_{\mu\nu}{}^a + (\omega\wedge\Sigma)_{\mu\nu}{}^a = \partial_\mu \Sigma_\nu{}^a - \partial_\nu \Sigma_\mu{}^a + \omega_\mu{}^a{}_b \Sigma_\nu{}^b - \omega_\nu{}^a{}_b \Sigma_\mu{}^b$

Transforms as a tensor under GCT for $\mu\nu$ and LLT for a

Proof Sketch: $\partial_\mu \Sigma_\nu{}^{a'} - \partial_\nu \Sigma_\mu{}^{a'} + \omega_\mu{}^{a'}{}_{b'} \underbrace{\boxed{\Sigma_\nu{}^{b'}}}_{*} - \omega_\nu{}^{a'}{}_{b'} \underbrace{\boxed{\Sigma_\mu{}^{b'}}}_{**} =$

oops, forgot to write these

use Thm at top of page.

$$= \partial_\mu(\Lambda^{a'}{}_a \Sigma_\nu{}^a) - \partial_\nu(\Lambda^{a'}{}_a \Sigma_\mu{}^a) - \Lambda^{a'}{}_a \Lambda^b{}_{b'} \omega_\nu{}^a{}_b{}^{(**)} + \Lambda^{a'}{}_a \Lambda^b{}_{b'} \omega_\mu{}^a{}_b{}^{(*)} + \Lambda^c{}_{b'} \partial_\nu \Lambda^{a'}{}_c{}^{(**)} - \Lambda^c{}_{b'} \partial_\mu \Lambda^{a'}{}_c{}^{(*)}$$

$$= \Lambda^{a'}{}_a(\partial_\mu \Sigma_\nu{}^a - \partial_\nu \Sigma_\mu{}^a) + \cancel{(\partial_\mu \Lambda^{a'}{}_a)\Sigma_\nu{}^a} - \cancel{(\partial_\nu \Lambda^{a'}{}_a)\Sigma_\mu{}^a} + \cancel{(\partial_\nu \Lambda^{a'}{}_c)\Lambda^c{}_{b'}\Sigma_\mu{}^{b'}}$$
$$+ \Lambda^{a'}{}_a(-\Lambda^b{}_{b'} \omega_\nu{}^a{}_b \Sigma_\mu{}^{b'} + \Lambda^b{}_{b'} \omega_\mu{}^a{}_b \Sigma_\nu{}^{b'}) - \cancel{(\partial_\mu \Lambda^{a'}{}_c)\Lambda^c{}_{b'}\Sigma_\nu{}^{b'}}$$

$$= \Lambda^{a'}{}_a(\partial_\mu \Sigma_\nu{}^a - \partial_\nu \Sigma_\mu{}^a + \omega_\mu{}^a{}_b \Sigma_\nu{}^b - \omega_\nu{}^a{}_b \Sigma_\mu{}^b).$$ //

Def^{n}/ $(\omega \wedge \Sigma)_{\mu\nu}{}^{a} = \omega_{\mu}{}^{a}{}_{b}\, \Sigma_{\nu}{}^{b} - \omega_{\nu}{}^{a}{}_{b}\, \Sigma_{\mu}{}^{b}$

wedge product of spin-connection with vector-valued one-form.

We should apply this formalism to <u>torsion</u> and <u>curvature</u>.

$T_{\mu\nu}{}^{a}$ vector-valued two form

$R^{a}{}_{b\mu\nu}$ (1,1) - tensor valued two form.

<u>Notation</u>

$$\theta^{a} = e^{a}, \quad \theta^{\mu} = dx^{\mu}$$

$$e^{a} = e_{\mu}{}^{a}\, dx^{\mu} \quad \text{and}$$

$$\boxed{\omega^{a}{}_{b} \overset{\text{def}^{n}}{=} \omega_{\mu}{}^{a}{}_{b}\, dx^{\mu}}$$

$$\left.\begin{aligned} &\boxed{T^{a} = de^{a} + \omega^{a}{}_{b} \wedge e^{b}} \\ &\boxed{R^{a}{}_{b} = d\omega^{a}{}_{b} + \omega^{a}{}_{c} \wedge \omega^{c}{}_{b}} \end{aligned}\right\} \text{“CARTAN'S STRUCTURE EQUATIONS”}$$

Consider,

$$\begin{aligned} T_{\mu\nu}{}^{\lambda} &= e^{\lambda}{}_{a}\, T_{\mu\nu}{}^{a} \\ &= e^{\lambda}{}_{a}\left(\partial_{\mu} e_{\nu}{}^{a} - \partial_{\nu} e_{\mu}{}^{a} + \omega_{\mu}{}^{a}{}_{b}\, e_{\nu}{}^{b} - \omega_{\nu}{}^{a}{}_{b}\, e_{\mu}{}^{b}\right) \\ &= e^{\lambda}{}_{a}\, \partial_{\mu} e_{\nu}{}^{a} + e^{\lambda}{}_{a}\, e_{\nu}{}^{b}\, \omega_{\mu}{}^{a}{}_{b} - \left(e^{\lambda}{}_{a}\, \partial_{\nu} e_{\mu}{}^{a} + e^{\lambda}{}_{a}\, e_{\mu}{}^{b}\, \omega_{\nu}{}^{a}{}_{b}\right) \\ &= \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu} \end{aligned}$$

(see top of pg. 9, we derived on 8)

$$\boxed{\begin{aligned} &\text{Th}^{m}/\; dT^{a} + \omega^{a}{}_{b} \wedge T^{b} = R^{a}{}_{b} \wedge e^{b} \\ &dR^{a}{}_{b} + \omega^{a}{}_{c} \wedge R^{c}{}_{b} - R^{a}{}_{c} \wedge \omega^{c}{}_{b} = 0 \end{aligned}}$$

Bianchi Identities.

<u>Proof</u>: differentiate CARTAN'S STRUCTURE EQ^{n}'S and use $d^{2} = 0$.

Deriving Bianchi Identities with exterior calculus

Given: $T^a = de^a + \omega^a{}_b \wedge e^b$ ☆ & $R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$ *

Differentiate, remember $d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^{|\alpha|}\alpha\wedge d\beta$ Graded Leibniz Rule

swap indices relable... solved ☆ for de^b

$$dT^a = \cancel{d(de^a)}^{\,0} + d\omega^a{}_b \wedge e^b - \omega^a{}_b \wedge de^b$$

$$d\omega^a{}_b = R^a{}_b - \omega^a{}_c \wedge \omega^c{}_b \quad \text{solving * for } d\omega^a{}_b$$

$$dT^a = (R^a{}_b - \omega^a{}_c \wedge \omega^c{}_b)\wedge e^b - \omega^a{}_b \wedge (T^b - \omega^b{}_{b'} \wedge e^{b'})$$

$$dT^a = R^a{}_b \wedge e^b - \cancel{\omega^a{}_c \wedge \omega^c{}_b \wedge e^b} - \omega^a{}_b \wedge T^b + \cancel{\omega^a{}_b \wedge \omega^b{}_c \wedge e^c}$$

$$\boxed{dT^a + \omega^a{}_b \wedge T^b = R^a{}_b \wedge e^b}$$

Differentiate *

$$dR^a{}_b = d(d\omega^a{}_b) + (d\omega^a{}_c)\wedge\omega^c{}_b - \omega^a{}_c \wedge (d\omega^c{}_b)$$

$$dR^a{}_b = (R^a{}_c - \omega^a{}_{a'} \wedge \omega^{a'}{}_c)\wedge\omega^c{}_b - \omega^a{}_c \wedge (R^c{}_b - \omega^c{}_{a'} \wedge \omega^{a'}{}_b)$$

$$dR^a{}_b = R^a{}_c \wedge \omega^c{}_b - \omega^a{}_c \wedge R^c{}_b$$

$$\boxed{dR^a{}_b + \omega^a{}_c R^c{}_b - R^a{}_c \wedge \omega^c{}_b = 0}$$

I'll follow Carroll and resist urge to further look at covariant der. at moment. Notice we have yet to impose torsion-free requirement or metric compatibility of the connection.

$$T = 0 \longrightarrow T^a = de^a + \omega^a{}_b \wedge e^b = 0$$

$$\therefore \boxed{de^a = -\omega^a{}_b \wedge e^b}$$ torsion free condition

$$\nabla g = 0 \longrightarrow \nabla_\mu \eta_{ab} = \cancelto{0}{\partial_\mu \eta_{ab}} - \omega_\mu{}^c{}_a \eta_{cb} - \omega_\mu{}^c{}_b \eta_{ac} = 0$$

$$\Rightarrow \omega_\mu{}^c{}_a \eta_{cb} = -\omega_\mu{}^c{}_b \eta_{ac}$$

$$\Rightarrow \boxed{\omega_{\mu ba} = -\omega_{\mu ab}}$$

<u>EXAMPLE: SPATIALLY FLAT, EXPANDING UNIVERSE</u>

$$ds^2 = \underbrace{-dt^2 + a^2(t)\delta_{ij}\,dx^i dx^j = \eta_{ab}\, e^a \otimes e^b}$$

$$e^0 = dt \qquad de^0 = d(dt) = 0 = -\omega^0{}_b \wedge e^b$$

$$e^i = a\,dx^i \qquad de^i = da \wedge dx^i = -\omega^i{}_b \wedge e^b$$

Now, $da = (\partial_\mu a)\,dx^\mu = \dot{a}\,dt$ since $\partial_i a = 0$ as a is fnct. of t alone.

$$0 = -\omega^0{}_b \wedge e^b \quad \& \quad \dot{a}\,dt \wedge dx^i = -\omega^i{}_b \wedge e^b$$
$$= -\omega^i{}_0 \wedge dt - \omega^i{}_j \wedge (dx^j a)$$

$e^0 = dt$

$e^i = a\,dx^i$

$a = a(t)$

$$\underbrace{\omega_{\mu ba} = -\omega_{\mu ab}}_{\text{metric compt. condition}} \to \begin{cases} \omega_{00} = 0 \\ \omega_{ij} = -\omega_{ji} \\ \omega_{0j} = -\omega_{j0} \end{cases}$$

Raising index; $\omega^a{}_b = \eta^{ac}\omega_{cb}$

$\omega^0{}_0 = \eta^{0c}\omega_{c0} = -\omega_{00} = 0 \quad \therefore \boxed{\omega^0{}_0 = 0}$

$\omega^0{}_j = \eta^{0c}\omega_{cj} = -\omega_{0j}$

$\omega^j{}_0 = \eta^{jc}\omega_{c0} = \omega_{j0} = -\omega_{0j}$ $\quad \boxed{\omega^0{}_j = \omega^j{}_0}$

$\omega^i{}_j = \eta^{ic}\omega_{cj} = \omega_{ij}$

$\omega^j{}_i = \eta^{jc}\omega_{ci} = \omega_{ji} = -\omega_{ij}$ $\quad \boxed{\omega^i{}_j = -\omega^j{}_i}$

$$\boxed{de^a = -\omega^a{}_b \wedge e^b}$$

$de^0 = d(dt) = 0 = -\omega^0{}_b \wedge e^b \Rightarrow 0 = -\omega^0{}_j \wedge (a\,dx^j)$ ①

$de^i = da \wedge dx^i = -\omega^i{}_b \wedge e^b \Rightarrow \dot{a}\,dt \wedge dx^i = -\omega^i{}_0 \wedge dt - \omega^i{}_j \wedge (a\,dx^j)$ ②

From ① $\omega^0{}_j = A\,dt + A_i\,dx^i \Rightarrow (A\,dt + A_i\,dx^i) \wedge (dx^j) = A\,dt \wedge dx^j + A_i\,dx^i \wedge dx^j = 0$

$\Rightarrow A = 0$ and $A_i = 0$ for $i \neq j$

Consequently only A_j nontrivial, $\omega^0{}_j = A_j\,dx^j$ plug into ②

$\dot{a}\,dt \wedge dx^i = -A_i\,dx^i \wedge dt - \omega^i{}_j \wedge (a\,dx^j)$

$= A_i\,dt \wedge dx^i - a\,\omega^i{}_j \wedge dx^j \Rightarrow \underbrace{A_i = \dot{a} \text{ and } \omega^i{}_j = 0}_{\text{possible solution.}}$

We found, $\underbrace{\omega^0{}_0 = 0, \quad \omega^0{}_j = \dot{a}\,dx^j = \omega^j{}_0, \quad \omega^i{}_j = 0}_{\text{Spin connection for } ds^2 = -dt^2 + a^2(t)\,\delta_{ij}\,dx^i \otimes dx^j}$

We wish to calculate,

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$$

Hence calculate:

$$d\omega^0{}_0 = 0$$
$$d\omega^0{}_j = d\dot{a} \wedge dx^j = \ddot{a}\,dt \wedge dx^j = d\omega^j{}_0$$
$$d\omega^i{}_j = 0$$

$$\omega^0{}_c \wedge \omega^c{}_0 = 0$$
$$\omega^i{}_c \wedge \omega^c{}_0 = 0$$
$$\omega^i{}_c \wedge \omega^c{}_j = \dot{a}^2\,dx^i \wedge dx^j$$

Therefore,

$$\boxed{\begin{aligned} R^0{}_0 &= 0 \\ R^0{}_j &= \ddot{a}\,dt \wedge dx^j \\ R^i{}_0 &= \ddot{a}\,dt \wedge dx^i \\ R^i{}_j &= \dot{a}^2\,dx^i \wedge dx^j \end{aligned}}$$

To compare with earlier work need to convert to curved index quantities,

$$R^\rho{}_{\sigma\mu\nu} = e^\rho{}_a\, e_\sigma{}^b\, R^a{}_{b\mu\nu}$$

$$e^\mu(e_b) = dx^\mu(e_b) = e^\mu{}_b \quad \& \quad e^a(\partial_\nu) = e_\nu{}^a$$

Alternatively, $dx^\mu = e^\mu{}_a\, e^a$ and $e^a = e_\mu{}^a\, dx^\mu$

$$\underbrace{dx^\mu = e^\mu{}_0\, dt + e^\mu{}_i\, (a\,dx^i)}_{(e^\mu{}_a) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/a & 0 & 0 \\ 0 & 0 & 1/a & 0 \\ 0 & 0 & 0 & 1/a \end{bmatrix}} \quad \text{and} \quad \underbrace{e^a = e_0{}^a\, dt + e_i{}^a\, dx^i}_{(e_\mu{}^a) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix}}$$

$$\underline{\underline{R}}^\rho{}_{\sigma\mu\nu} = e^\rho{}_a\, e_\sigma{}^b\, R^a{}_{b\mu\nu}$$

Remark: the μ vs a notation breaks down here, so I added the $=$ to distinguish.

$$R^0{}_0 = 0$$
$$R^0{}_j = \ddot{a}\, dt \wedge dx^j$$
$$R^i{}_0 = \ddot{a}\, dt \wedge dx^i$$
$$R^i{}_j = \dot{a}^2\, dx^i \wedge dx^j$$

$$R^0{}_{0\mu\nu} = 0$$
$$R^0{}_{j\mu\nu} = \ddot{a}(\delta_\mu^0 \delta_\nu^j - \delta_\nu^0 \delta_\mu^j)$$
$$R^i{}_{0\mu\nu} = \ddot{a}(\delta_\mu^0 \delta_\nu^i - \delta_\nu^0 \delta_\mu^i)$$
$$R^i{}_{j\mu\nu} = \dot{a}^2(\delta_\mu^i \delta_\nu^j - \delta_\nu^i \delta_\mu^j)$$

$$\therefore\ R^0{}_{j0l} = \ddot{a}(\delta_0^0 \delta_l^j - \delta_l^0 \delta_0^j) = \ddot{a}\,\delta_l^j$$

$$\underline{\underline{R}}^0{}_{j0l} = e^0{}_a\, e_j{}^b\, R^a{}_{b0l} = a\, R^0{}_{j0l} = a\ddot{a}\,\delta_l^j$$

Continuing in the calculations from 16 we can show,

$$\underline{\underline{R}}^{i}{}_{0k0} = e^{i}{}_{a}\, e_{0}{}^{b}\, R^{a}{}_{bk0}$$
$$= \left(\tfrac{1}{a}\delta^{i}_{a}\right)\left(1\,\delta^{b}_{0}\right) R^{a}{}_{bk0}$$
$$= \frac{1}{a} R^{i}{}_{0k0}$$
$$= \frac{1}{a}\ddot{a}\left(\delta^{0}_{k}\delta^{i}_{0} - \delta^{0}_{0}\delta^{i}_{k}\right)$$
$$= \frac{-1}{a}\ddot{a}\,\delta^{i}_{k}$$

$$\underline{\underline{R}}^{i}{}_{jkl} = e^{i}{}_{a}\, e_{j}{}^{b}\, R^{a}{}_{bkl}$$
$$= \left(\tfrac{1}{a}\delta^{i}_{a}\right)\left(a\,\delta^{b}_{j}\right) R^{a}{}_{bkl}$$
$$= R^{i}{}_{jkl}$$
$$= \dot{a}^{2}\left(\delta^{i}_{k}\delta^{j}_{l} - \delta^{i}_{l}\delta^{j}_{k}\right)$$

Remark: see pg. 333 in Carroll and set $K = 0$

Def^n/ as before, $R_{\sigma\nu} = R^{\lambda}{}_{\sigma\lambda\nu}$ defines the Ricci Tensor. Hence

$$R_{00} = R^{\lambda}{}_{0\lambda 0} = \frac{-\ddot{a}}{a}\delta^{i}_{i} = \frac{-3\ddot{a}}{a} \quad \& \quad R_{i0} = R^{\lambda}{}_{i\lambda 0} = 0$$

$$R_{ij} = R^{\lambda}{}_{i\lambda j} = R^{0}{}_{i0j} + R^{k}{}_{ikj} = a\ddot{a}\,\delta_{ij} + \dot{a}^{2}\left(3\delta_{ij} - \delta_{ij}\right)$$