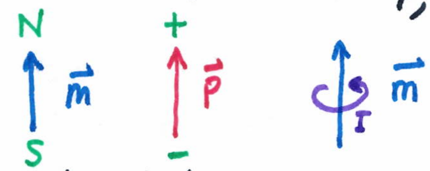
LECTURE 21 : MAGNETISM IN MATTER

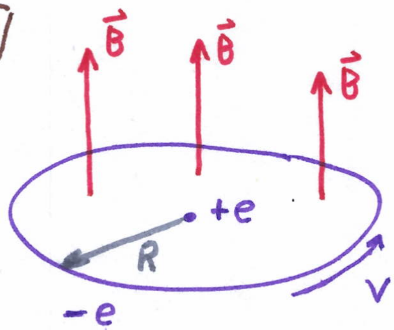
The net-force on a current loop is zero when $\vec{B}$ is <u>uniform</u> over the loop. However, when $\vec{B}$ is not uniform, for an infintesimal loop,

$$\boxed{\vec{F} = \nabla(\vec{m}\cdot\vec{B})}$$



this agrees with the corresponding electrostatic result for dipole $\vec{p}$, $\vec{F} = \nabla(\vec{p}\cdot\vec{E})$

E1



$$\Delta v = \frac{eRB}{2m_e}$$

(pages 273 – 274 of GRIFFITHS has details)

"when B turns on the electon speeds up"

$$\Delta\vec{m} = \frac{-1}{2}e(\Delta v)R\hat{z} = -\frac{e^2R^2}{4m_e}\vec{B}$$

<u>Key Point</u>: change in $\vec{m}$ is opposite the direction of $\vec{B}$

This atomic process is responsible for <u>diagamagnetism</u> ($\vec{M}$ antiparallel to $\vec{B}$)

Def^n/ $\vec{M} \equiv$ magnetic dipole moment per unit volume

Remark: $\vec{M}$ can come from

diamagnetism : (water, copper, gold, graphite, bismuth)

paramagnetism : (aluminum, platinum, liquid oxygen)

ferromagnetism : (iron, nickel, cobalt)

BOUND CURRENT DERIVATION:

Suppose we're given a material with magnetization $\vec{M}$ throughout some volume $\mathcal{V}$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{M}(\vec{r}\,') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'$$

Notice that $\nabla' = \hat{x}'\partial_{x'} + \hat{y}'\partial_{y'} + \hat{z}'\partial_{z'}$ gives $\nabla'\left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \left[\underline{\vec{M}(\vec{r}\,') \times \left(\nabla' \frac{1}{\mathscr{r}}\right)}\right] d\tau'$$

Then consider the product rule,

$$\nabla' \times \left(\frac{1}{\mathscr{r}}\vec{M}\right) = \frac{1}{\mathscr{r}}\nabla' \times \vec{M} - \vec{M} \times \nabla'\left(\frac{1}{\mathscr{r}}\right)$$

$$\underline{\vec{M}(\vec{r}\,') \times \left(\nabla' \frac{1}{\mathscr{r}}\right)} = \frac{1}{\mathscr{r}}\nabla' \times \vec{M}(\vec{r}\,') - \nabla' \times \left(\frac{1}{\mathscr{r}}\vec{M}(\vec{r}\,')\right)$$

Consequently,

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\left[\int \left(\frac{1}{\mathscr{r}}\nabla' \times \vec{M}(\vec{r}\,') - \nabla' \times \left(\frac{1}{\mathscr{r}}\vec{M}(\vec{r}\,')\right)\right) d\tau'\right]$$

$\downarrow$ hwk. 1.61b

$$= \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{\nabla' \times \vec{M}(\vec{r}\,')}{\mathscr{r}}\, d\tau' + \frac{\mu_0}{4\pi}\oint_{\partial\mathcal{V}} \frac{1}{\mathscr{r}}\left[\vec{M}(\vec{r}\,') \times d\vec{a}\,'\right]$$

Thus, Def$^{\underline{n}}$/ $\vec{J}_b = \nabla \times \vec{M}$ and $\vec{K}_b = \vec{M} \times \hat{n}$ where $\hat{n}$ is unit-normal to $\partial\mathcal{V}$

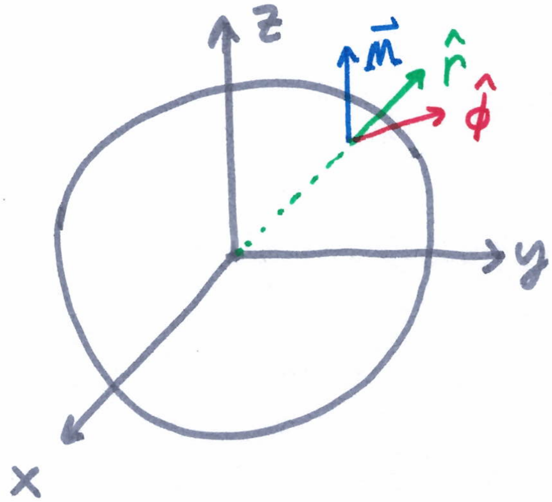
Yields,

$$\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{\vec{J}_b(\vec{r}\,')}{\mathscr{r}}\, d\tau' + \frac{\mu_0}{4\pi}\oint_{\partial\mathcal{V}} \frac{\vec{K}_b(\vec{r}\,')}{\mathscr{r}}\, da'}$$

E2 Uniformly magnetized sphere of radius R with magnetization $\vec{M} = M\hat{z}$

$$\vec{J}_b = \nabla \times \vec{M} = 0$$

$$\vec{K}_b = \vec{M} \times \hat{n} = M\,\hat{z} \times \hat{r}$$



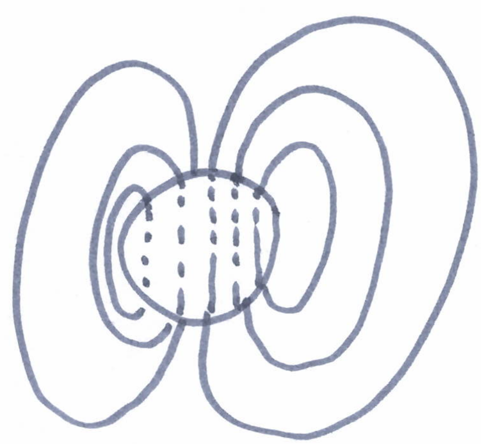
$$\hat{z} \times \hat{r} = \langle 0,0,1 \rangle \times \langle \cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta \rangle$$
$$= \langle -\sin\phi \sin\theta, \cos\phi \sin\theta, 0 \rangle$$
$$= \sin\theta \langle -\sin\phi, \cos\phi, 0 \rangle$$
$$= \underline{\sin\theta\, \hat{\phi}}$$

$$\underbrace{\vec{K} = \sigma \vec{v} = \sigma \omega R \sin\theta\, \hat{\phi}}_{\text{example from LECTURE 20}}$$

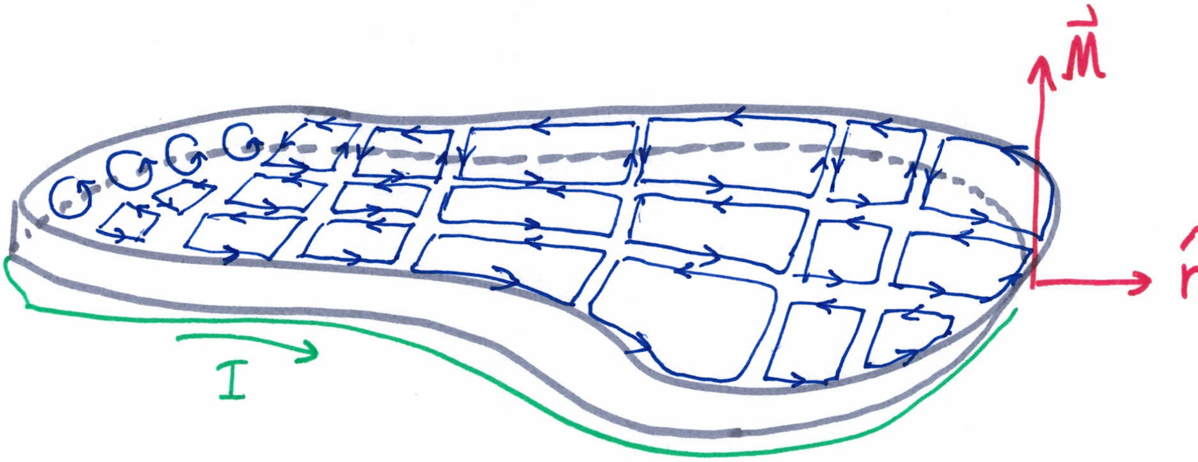
So, $\vec{K}_b = M \sin\theta\, \hat{\phi}$ $\Rightarrow$ substitute $M = \sigma \omega R$ into our earlier work,

$$\vec{B} = \frac{2}{3}\mu_0 \sigma R \omega \hat{z} = \frac{2}{3}\mu_0 M \hat{z}$$

$$\therefore \boxed{\vec{B} = \frac{2}{3}\mu_0 \vec{M}}$$



Physical Derivation of Bound Currents



- interior currents cancel out
  just left with ribbon current on edge

$$\underline{\vec{K}_b = \vec{M} \times \hat{n}}.$$

- for nonuniform magnetization can derive
  (see p. 280 – 281)

$$\underline{\vec{J}_b = \nabla \times \vec{M}}.$$

<u>Remark</u>: $\nabla \cdot \vec{J}_b = \nabla \cdot (\nabla \times \vec{M}) = 0$

the bound current is a steady current.

The Auxiliary Field $\vec{H}$

Given a material with magnetization $\vec{M}$ we generally have both bound currents ($\vec{J}_b$) & $\vec{K}_b$) associated to $\vec{M}$ via $\vec{J}_b = \nabla \times \vec{M}$ and $\vec{K}_b = \vec{M} \times \hat{n}$ and free currents ($\vec{J}_f$, $\vec{K}_f$ etc)

$$\boxed{\vec{J} = \vec{J}_b + \vec{J}_f}$$ (total current density)

Ampere's Law for this steady current,

$$\nabla \times \vec{B} = \mu_0 \vec{J} = \mu_0 (\nabla \times \vec{M} + \vec{J}_f)$$

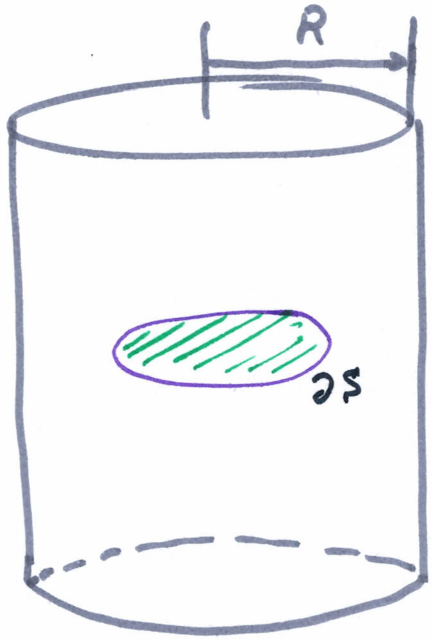
$$\therefore \quad \nabla \times \left(\frac{1}{\mu_0} \vec{B} - \vec{M} \right) = \vec{J}_f$$

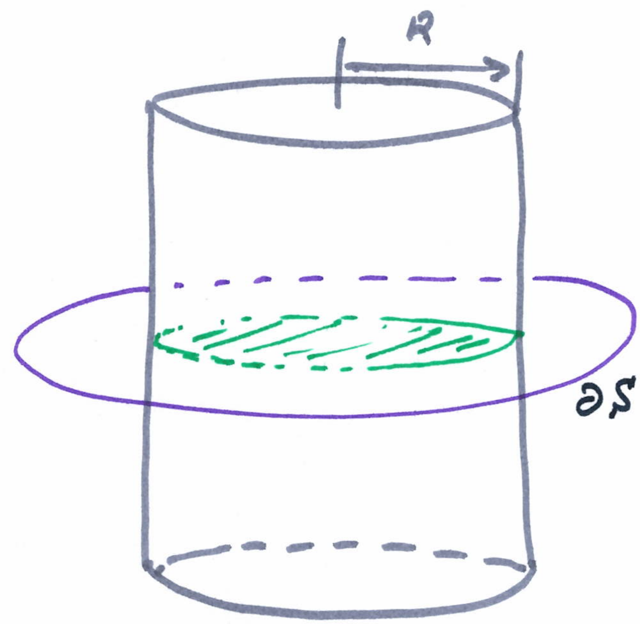
$$\boxed{\text{Def}^{\underline{n}}/ \quad \vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M}}$$ (the "auxiliary" field, oh honestly we just call it "H")

Remark: some texts call $\vec{H}$ the "magnetic field" I will not do this (on purpose at least).

Ampere's Law $\Rightarrow$
$$\boxed{\begin{gathered} \nabla \times \vec{H} = \vec{J}_f \\ \oint_{\partial S} H \cdot d\vec{l} = \int_S \vec{J}_f \cdot d\vec{a} = I_{enclosed} \end{gathered}}$$

E3 A long copper rod of radius R carries uniform free current I. Find $\vec{H}$ inside and outside the rod.





$$I_{enc} = \left(\frac{\pi s^2}{\pi R^2}\right) I$$

$$\oint_{\partial S} \vec{H} \cdot d\vec{l} = \frac{s^2 I}{R^2}$$

$$H(2\pi s) = \frac{s^2 I}{R^2}$$

$$\boxed{\vec{H} = \frac{sI}{2\pi R^2}\hat{\phi}}$$

$0 \leq s \leq R$

$$I_{enc} = I$$

$$\oint_{\partial S} \vec{H} \cdot d\vec{l} = I$$

$$H(2\pi s) = I$$

$$\boxed{\vec{H} = \frac{I}{2\pi s}\hat{\phi}}$$

$s \geq R$

Outside, $\vec{M} = 0$ hence $\vec{H} = \frac{1}{\mu_0}\vec{B}$ and we deduce $\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$ for $s \geq R$. At the moment, $\vec{B} = ?$ for $s < R$ since $\vec{M}$ has not been specified.

Linear & Nonlinear Media

For many substances the magnetization is proportional to the magnetic field.

$$\boxed{\text{Def}^n/\ \vec{M} = \chi_m \vec{H}}$$ ← for Linear Media.

But, as we just defined, $\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$

hence $\vec{M} = \chi_m\left(\frac{1}{\mu_0}\vec{B} - \vec{M}\right)$ consequently

$$\begin{aligned}\vec{B} &= \mu_0\left(\vec{H} + \vec{M}\right)\\ &= \mu_0\left(\vec{H} + \chi_m \vec{H}\right)\\ &= \mu_0(1+\chi_m)\vec{H} \qquad \therefore \quad \vec{B} = \mu \vec{H}\end{aligned}$$

Def^n/ $\mu = \mu_0(1+\chi_m)$ gives permeability of the material

Remark: $\chi_m > 0 \Rightarrow$ paramagnetic.
$\chi_m < 0 \Rightarrow$ diamagnetic.

we'll conclude with a discussion of ferromagnetism (see p. 290 – 292 GRIFFITHS)