LECTURE 22 : ELECTRODYNAMICS & FARADAY'S LAW

We begin by studying the emf = electro-motive-force, in other words, we seek to explain how $\vec{J}$ arises within a given circuit. I'm following § 7.1 of GRIFFITHS 5th Ed.

$$\vec{J} = \sigma \vec{f}$$

σ — conductivity; $\vec{f}$ — force per unit charge

$$\rho = \frac{1}{\sigma} = \text{resistivity}$$

$\sigma \longrightarrow 0$ for perfect conductor

$\sigma \longrightarrow \infty$ for perfect insulator

For example, $\sigma = 1.59 \times 10^{-8}\ \Omega m$ for silver whereas $\sigma = 10^{9} - 10^{14}\ \Omega m$ for glass

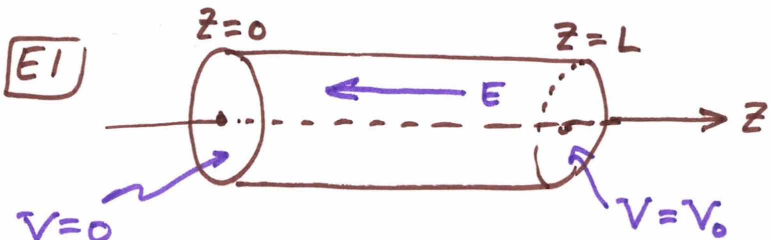
$$\vec{J} = \sigma(\vec{E} + \vec{v} \times \vec{B})$$

since v small typically, however, in Plasmas...

$$\boxed{\vec{J} = \sigma \vec{E}} \quad \text{(Ohm's Law)}$$

Remark: a very small $\vec{E}$-field is needed to move charge inside good conductors, we treat conductors as equipotentials (this assumes $\nabla V = 0$ hence $\vec{E} = -\nabla V = 0$)

E1



- cross-sectional area A
- length L
- conductivity σ
- potential fixed constant at each end

$$V(z) = \frac{V_o}{L} z$$

$$\vec{E} = -\nabla V = \frac{-V_o}{L}\hat{z}$$

Therefore, the electric field is uniform within the material and,

$$I = JA = \sigma E A = \frac{\sigma A}{L} V_o$$

Units for R are Ω = Ohms

$$\boxed{V_o = \left(\frac{L}{\sigma A}\right) I = I(\rho L/A) = IR} \quad \text{Ohm's Law}$$

Drude's classical model of conductivity gives

$$\boxed{\vec{J} = nfq\vec{v}_{ave} = \frac{nfq\lambda}{2V_{thermal}} \cdot \frac{\vec{F}}{m} = \left(\frac{nf\lambda q^2}{2mV_{thermal}}\right)\vec{E}}$$

where q is the charge moving, λ is distance between collisions for q in the lattice of the material, then $V_{thermal}$ is speed of q due to thermal (random vibration) and f = # of free electrons per molecule, m the mass of the charge. Notice (I think this is the "drift" velocity)

$$\boxed{V_{ave} = \frac{a\lambda}{2V_{thermal}}}$$

Remark: as $V_{thermal}$ increases we see $\sigma = \dfrac{nf\lambda q^2}{2mV_{thermal}}$ decreases which means $\rho = \dfrac{1}{\sigma}$ increases, so resistance increases with temperature.

WHY IS CURRENT I SAME AROUND A LOOP?

$$\vec{f} = \vec{f}_s + \vec{E}$$

$q\vec{f}_s$ = force from source on q

$q\vec{E}$ = electrostatic force on q

$$\boxed{\text{Def}^{n}/ \quad \mathcal{E} = \oint \vec{f} \cdot d\vec{l} = \oint \vec{f}_s \cdot d\vec{l}}$$

$\mathcal{E}$ is the emf the so-called electromotive force. It's actually a force per unit charge's integral around the loop.

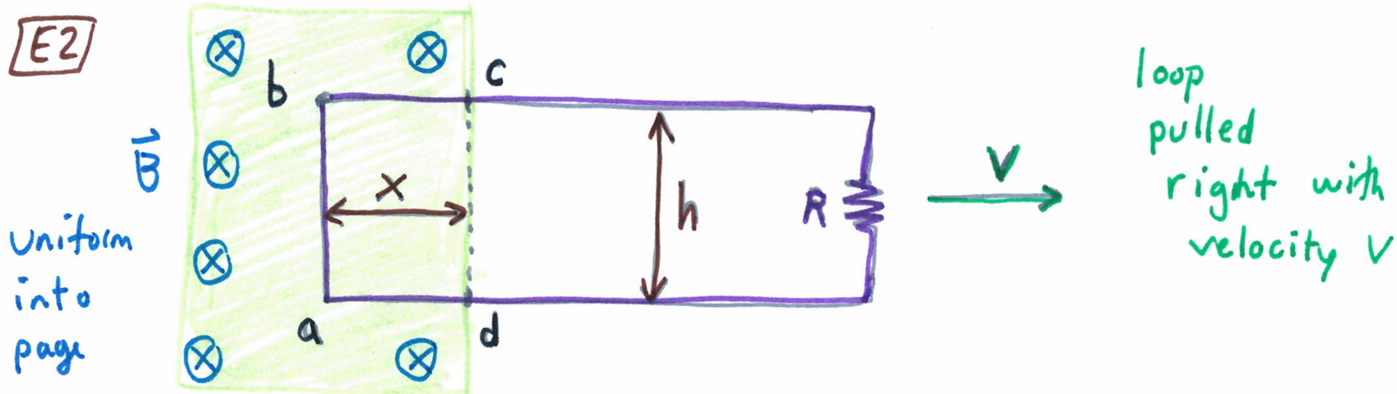
(note $\oint \vec{E} \cdot d\vec{l} = 0$ since $\vec{E}$ is conservative)

FOOTNOTE: "current in an electric circuit is somewhat analogous to flow of water in a closed system of pipes, with gravity playing role of the electrostatic field, and a pump (lifting the water up against gravity) in the role of the battery. Here height is analogous to voltage.

MOTIONAL EMF:

Perhaps the most common method to generate EMF is to move a wire through a magnetic field

E2



- charges in ab experience magnetic force qvB pushes charge around loop in CW-fashion
- force on bc and ad charges cancels out so only the ab-segment need be considered

$$\mathcal{E} = \oint \vec{f}_{mag} \cdot d\vec{l} = vBh$$

(I believe this assumes CW-orientated loop)

- How to understand the physics here?
  - charges in ab have velocity $\vec{W} = \langle v, u, 0 \rangle$ once current flowing then force per unit charge is:

$$\vec{f}_{mag} = \vec{W} \times \vec{B} = \langle v, u, 0 \rangle \times \langle 0, 0, -B \rangle$$

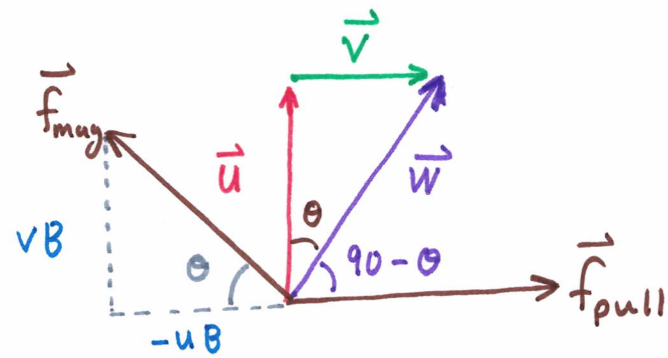
$$\vec{f}_{mag} = \langle -uB, vB, 0 \rangle$$

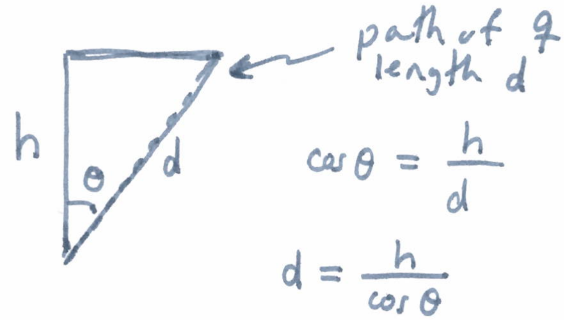
Then magnetic force has component $-quB\hat{x}$ on q in the ab segment. Thus $\vec{f}_{pull} = uB\hat{x}$ must be exerted to obtain constant velocity $\vec{v}$ for loop.

E2 continuing,

charge q in ab-segment moves with net-velocity
$\vec{w} = \langle v, u, 0 \rangle$

$$\int \vec{f}_{pull} \cdot d\vec{l} = (uB)\left(\frac{h}{\cos\theta}\right)\sin\theta = vBh = \mathcal{E}$$





($d\vec{l}$ is along $\vec{w}$ which traces out path travelled by q)
notice $\cos(90-\theta) = \sin\theta$.

Remark: the emf is due to work done by $\vec{f}_{pull}$
Notice, $\vec{f}_{mag} \perp d\vec{l}$ does no work.

There is a nicer method to calculate EMF. Let Φ_B be the magnetic flux through the loop; $\Phi_B = \int \vec{B} \cdot d\vec{a}$

$$\Phi_B = Bhx$$

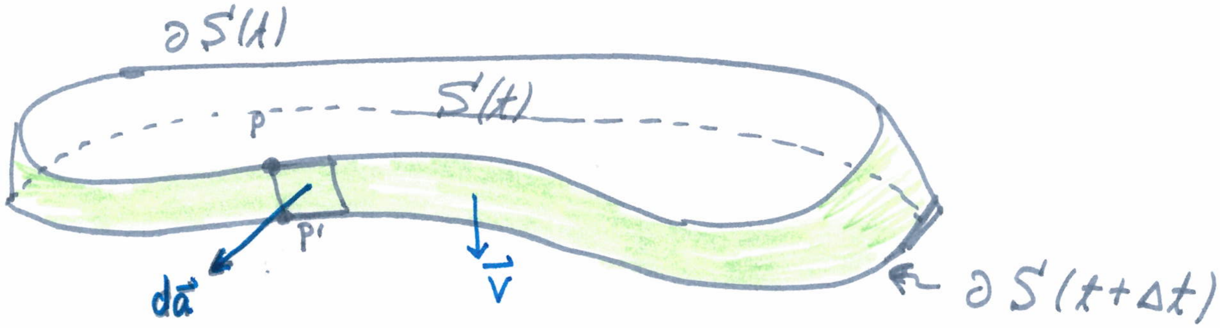
Then observe that $-v = \frac{dx}{dt}$ and as B and h are constant,

$$\frac{d\Phi}{dt} = Bh\frac{dx}{dt} = -Bhv$$

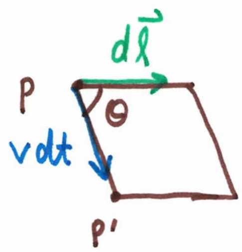
$$\boxed{\mathcal{E} = -\frac{d\Phi}{dt}}$$ the flux rule for motional emf

FLUX RULE FOR MOTIONAL EMF

Th^m/ Let S be a surface bounded by a wire and let $\Phi = \int_S \vec{B}\cdot d\vec{a}$ then the motional emf generated by $\vec{B}$ in the wire is given by $\mathcal{E} = -\dfrac{d\Phi}{dt}$



$$d\Phi = \Phi(t+dt) - \Phi(t) = \Phi_{ribbon} = \int_{ribbon} \vec{B}\cdot d\vec{a}$$



$$d\vec{a} = (\vec{V}\times d\vec{l})\,dt$$

$\vec{W} = \vec{u} + \vec{V}$ note $\vec{u} \parallel d\vec{l}$

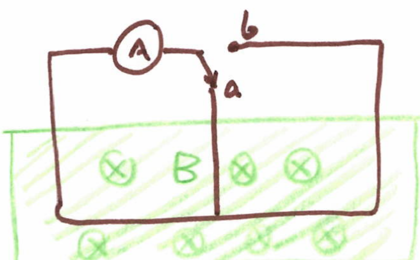
$$\frac{d\Phi}{dt} = \oint_R \vec{B}\cdot(\vec{V}\times d\vec{l}) = \oint_R \vec{B}\cdot(\vec{W}\times d\vec{l})$$

$$= \oint d\vec{l}\cdot(\vec{B}\times\vec{W})$$

$$= -\oint d\vec{l}\cdot(\vec{W}\times\vec{B})$$

$$= -\oint d\vec{l}\cdot\vec{f}_{mag} = -\mathcal{E}. \;//$$

E3 Beware this works for loops.



What happens when we close the switch from $a \longrightarrow b$?

E4 See Example 7.4 not sure we have time for it today, but it's very nice.

Remark: eddy currents

FARADAY'S LAW

Outside of the discussion in GRIFFITHS I would usually say that Faraday's Law is simply

$$\boxed{\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\left[\int_S \vec{B}\cdot d\vec{a}\right]}$$

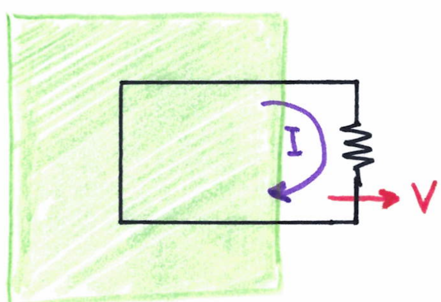
where $\mathcal{E}$ is change in potential developed around ∂S

GRIFFITHS CALLS THIS THE "UNIVERSAL FLUX RULE"

GRIFFITHS sometimes agrees, but he wants to draw a distinction between motional emf vs. problems where the induced electric field account for $\mathcal{E} = \int_{\partial S} \vec{E}\cdot d\vec{l}$

I'll follow his account of Faraday's experiments,

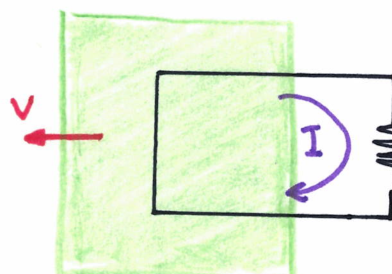
EXPERIMENT 1



$\vec{B}$ into page

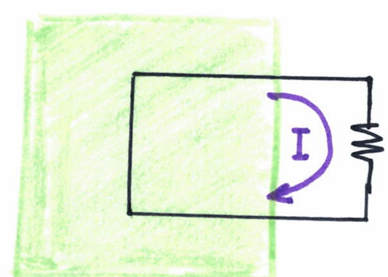
As with $\boxed{E2}$ the emf generated by motional emf

EXPERIMENT 2



$\vec{B}$ into page

EXPERIMENT 3



$\vec{B}$ varying with time

induced $\vec{E}$ – field creates emf

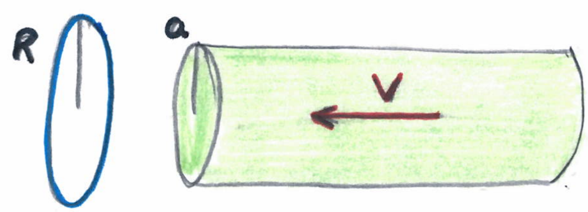
Faraday found the following rule relating changing B-field to an induced E-field:

$$\mathcal{E} = \oint_{\partial S} \vec{E}\cdot d\vec{l} = -\frac{d\Phi}{dt} = \frac{-d}{dt}\int_S \vec{B}\cdot d\vec{a}$$

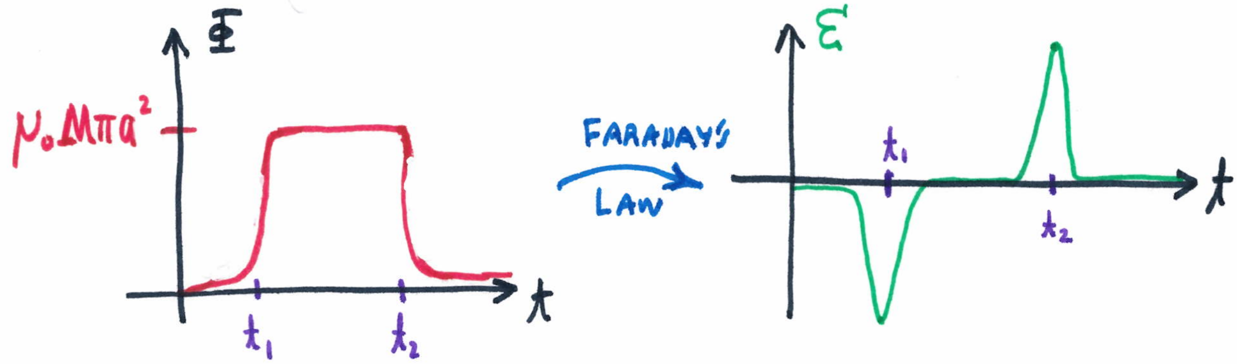
Then $\displaystyle\int_S (\nabla\times\vec{E})\cdot d\vec{a} = -\int_S \frac{\partial\vec{B}}{\partial t}\cdot d\vec{a} \Rightarrow \boxed{\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}}$

$\boxed{E5}$ A long cylindrical magnet of length L with radius a carries a uniform magnetization M parallel to its axis. Suppose it passes through ring at constant velocity v through slightly larger ring R then graph the emf

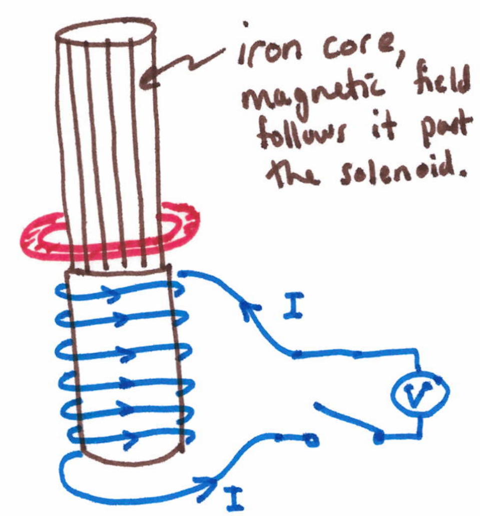
$a < R$



$$\mathcal{E} = -\frac{d\Phi}{dt}$$



$\boxed{E6}$



(THOMSON'S JUMPING RING EXPERIMENT)

when close switch a current flows through solenoid hence B-field is established in the ring $\Rightarrow$ induced current goes opposite direction and then, sometimes, the ring jumps.

The Induced Electric Field:

Faraday's Law generalizes $\nabla \times \vec{E} = 0$ into the time-dependent context. Now we face,

$$\nabla \cdot \vec{E} = \rho/\varepsilon_0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

<u>A pure Faraday field</u> arises from $\vec{B}$ changing alone $(\rho = 0)$

$$\underbrace{\begin{matrix} \nabla \cdot \vec{E} = 0 & & \nabla \cdot \vec{B} = 0 \\ & \longleftrightarrow & \\ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} & & \nabla \times \vec{B} = \mu_0 \vec{J} \end{matrix}}$$

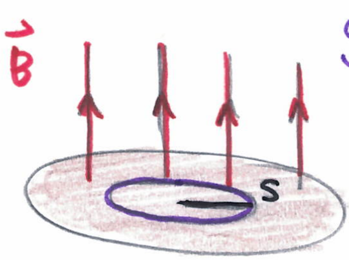
we can treat $-\frac{\partial \vec{B}}{\partial t}$ as if it was a current and calculate the induced $\vec{E}$ field in light of the $\longleftrightarrow$.

$$\vec{E} = \frac{-1}{4\pi} \int \frac{\frac{\partial \vec{B}}{\partial t} \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'$$

E7 A uniform magnetic field $\vec{B} = B(t)\hat{z}$ fills the region $0 \leq s \leq R$. Find the induced $\vec{E}$ field.

Calculate $\frac{\partial \vec{B}}{\partial t} = -\frac{dB}{dt}\hat{z} \Rightarrow \nabla \times \vec{E} = \left(\frac{dB}{dt}\right)\hat{z}$ which means the induced E field follows $\hat{\phi}$ direction (by analogy with the line current in $\hat{z}$-direction's $\vec{B}$-field)

$$\oint \vec{E} \cdot d\vec{l} = E(2\pi s) = -\frac{d}{dt}\left(\pi s^2 B(t)\right) = -\pi s^2 \frac{dB}{dt}$$



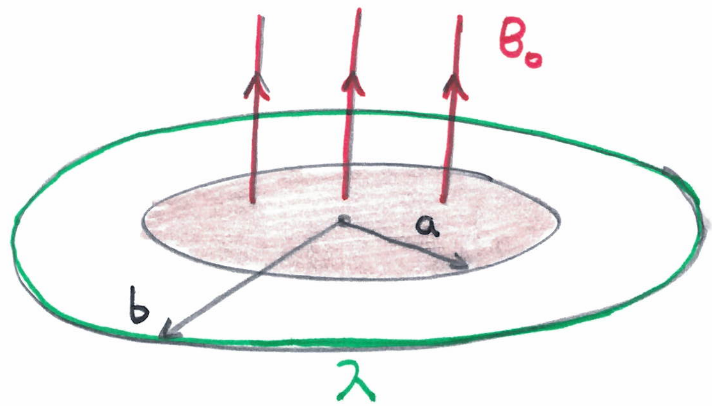
$$\boxed{\vec{E} = \frac{-s}{2}\frac{dB}{dt}\hat{\phi}}$$

If B is increasing then E field lines are CW

QUASISTATIC APPROXIMATION

We're using the techniques of <u>magnetostatics</u> when the magnetic field is changing, this is ok in as much as the magnetic field is not changing too rapidly (this claim we must trust for now) we study it in the radiation chapter

E8



If we switch off the magnetic field B_o then the flux through the ring with charge λ changes. Apply Faraday's Law at radius b,

$$\oint \vec{E} \cdot d\vec{l} = E(2\pi b) = -\frac{d\Phi}{dt} = -\pi a^2 \frac{dB}{dt}$$

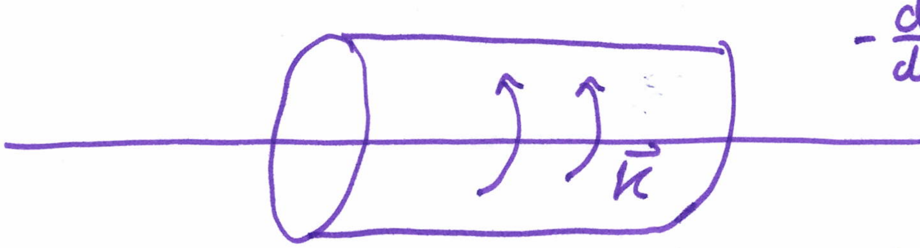
$$\boxed{\vec{E} = -\frac{a^2}{2b}\frac{dB}{dt}\hat{\phi}}$$

<u>Remark</u>: Griffiths shows the <u>angular</u> <u>momentum</u> imparted to the charged ring λ is given by $\Delta\vec{L} = \lambda\pi a^2 b B_o$ irrespective of how fast or slow we switch off B_o. This begs a question ... where did the angular momentum come from?

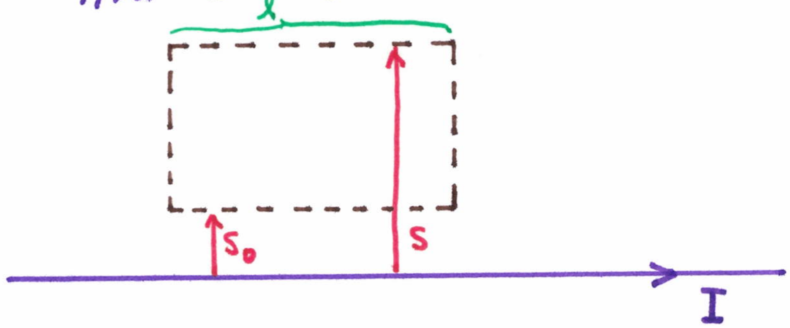
E9 An infinitely long straight wire carries a slowly varying current $I(t)$. Determine the induced E-field as function of distance s from the wire.

Quasistatic assumption $\Rightarrow$ $\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$

$$-\frac{d\vec{B}}{dt} = \left(-\frac{\mu_0 \frac{dI}{dt}}{2\pi s}\right)\hat{\phi}$$



like current in solenoid wraps around $\hat{\phi}$ direction



$$\oint \vec{E}\cdot d\vec{l} = E(s_0)l - E(s)l = -\frac{d}{dt}\int \vec{B}\cdot d\vec{a}$$

$$= -\frac{d}{dt}\int \frac{\mu_0 I}{2\pi s}\hat{\phi}\cdot d\vec{a}$$

$$= -\frac{\mu_0}{2\pi}\left(\frac{dI}{dt}\right) l \int_{s_0}^{s} \frac{ds'}{s'}$$

$$= \frac{-\mu_0 l}{2\pi}\frac{dI}{dt}\left(\ln(s) - \ln(s_0)\right)$$

Remark: this derivation assumes changing $I(t)$ at $s = 0$ immediately causes B to change at $s = s_0$ for arbitrary s_0. That is a violation of Chapter [illegible] physics, which puts a speed limit on info spreading.

$$\boxed{\vec{E}(s) = \left[\frac{\mu_0}{2\pi}\frac{dI}{dt}\left(\ln(s) - \ln(s_0)\right) + E(s_0)\right]\hat{z}}$$

Notice $E \rightarrow \infty$ as $s \rightarrow \infty$, but is this correct? (No)