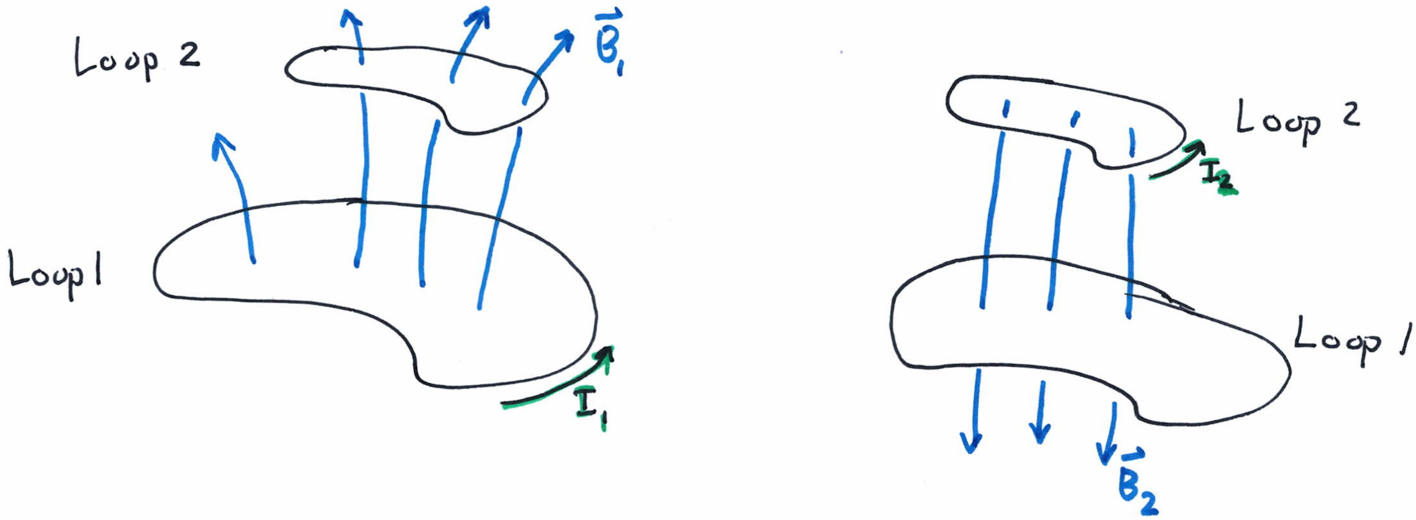
Lecture 23 : Induction, Energy in Magnetic Field, Maxwell's Equations



We consider two loops at rest.

① • running steady current I_1 through loop 1 generated magnetic field $\vec{B}_1$ which gives flux through Loop 2 (Φ_2)

② • running steady current I_2 through loop 2 generates mag. field $\vec{B}_2$ which gives flux through Loop 1.

We'll focus on scenario ①,

$$\vec{B}_1 = \frac{\mu_0}{4\pi} I_1 \oint_{loop\,1} \frac{d\vec{l}_1 \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

$\vec{B}_1$ proportional to I_1 thus M_{21} exists as we can pull I_1 out of flux integral

$$\Phi_2 = \int_{area\,2} \vec{B}_1 \cdot d\vec{a}_2 = \underbrace{M_{21}\, I_1}$$

Def^{n} of mutual induction

However, the vector potential may give us some additional insight,

$$\Phi_2 = \int_{area\,2} (\nabla \times \vec{A}_1) \cdot d\vec{a}_2 = \oint_{loop\,2} \vec{A}_1 \cdot d\vec{l}_2$$

Recall,

$$\vec{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint_{\text{loop} 1} \frac{d\vec{l}_1}{r}$$

Consequently,

$$\Phi_2 = \frac{\mu_0 I_1}{4\pi} \oint_{\text{loop } 2} \left(\oint_{\text{loop } 1} \frac{d\vec{l}_1}{r} \right) \cdot d\vec{l}_2 = M_{21} I_1$$

$$\boxed{M_{21} = \frac{\mu_0}{4\pi} \oint_{\text{loop } 2} \oint_{\text{loop } 1} \frac{d\vec{l}_1 \cdot d\vec{l}_2}{r}}$$

Neumann Formula

Remark: we find $M_{12} = M_{21}$ since the roles of loop 1 and 2 could be swapped and the formula above would give

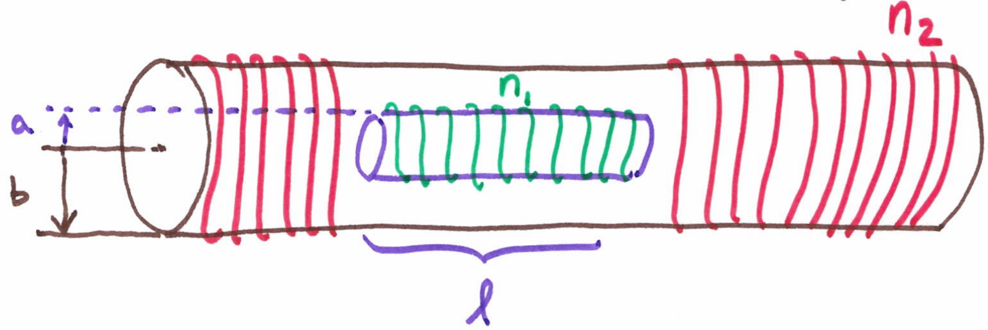
$$M_{12} = \frac{\mu_0}{4\pi} \oint_{\text{loop } 1} \oint_{\text{loop } 2} \frac{d\vec{l}_2 \cdot d\vec{l}_1}{r}$$

and since r is symmetric and $d\vec{l}_1 \cdot d\vec{l}_2 = d\vec{l}_2 \cdot d\vec{l}_1$ we

$r = \| \vec{r}_1 - \vec{r}_2 \| = \| \vec{r}_2 - \vec{r}_1 \|$

we find $M_{12} = M_{21}$

E1 A short solenoid of length ℓ and radius a with n_1 turns per length lies on axis of long solenoid with radius $b > a$ and n_2 turns per length. Suppose current I flows in short solenoid. What is flux through long solenoid? (Φ_2)



Reverse the problem. Let I flow through outside solenoid then $B_2 = \mu_0 I n_2$ by the usual arguments

and

$$\Phi_1 = (\pi a^2)(\underbrace{n_1 \ell}_{\#\text{ of loops in short solenoid}}) B_2 = I \underbrace{(\mu_0 n_1 n_2 \pi a^2 \ell)}_{M_{12}}$$

But then we know $M_{12} = M_{21}$ and

$$\boxed{\Phi_2 = M_{21} I = \mu_0 I n_1 n_2 \pi a^2 \ell}$$

If the current I changes slow enough for quasistatic approximation to hold then we have that

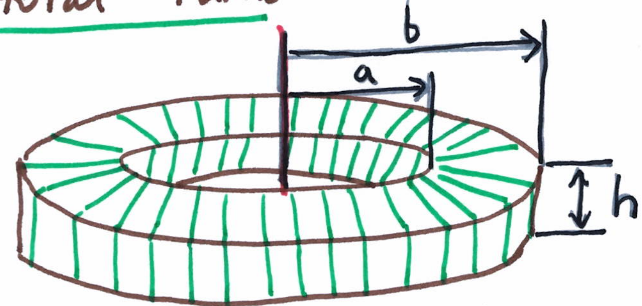
$$\boxed{\mathcal{E}_2 = -\frac{d\Phi_2}{dt} = -M \frac{dI_1}{dt}}$$

Moreover, even in a single loop if we change the magnetic flux then since $\underbrace{\Phi = LI}_{\text{implicitly defines self-inductance}}$ we find

$$\boxed{\mathcal{E} = -\frac{d\Phi}{dt} = -L\frac{dI}{dt}}$$

$[L] = \text{Henrie} = \frac{V}{s \cdot A}$

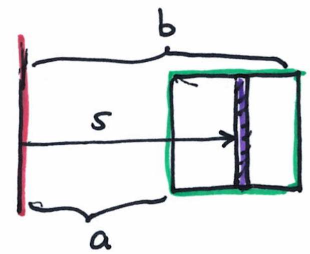
$\boxed{\text{E2}}$ Find the self-inductance of toroidal coil with rectangular cross-section with height h and inner radius a and outside radius b and N total turns



Ampere's Law and the usual arguments show that $\vec{B}$ within the toroid is given by

$$B = \frac{\mu_0 N I}{2\pi s}$$

at distance s from the center. Hence,



$$\Phi_1 = \int \vec{B}\cdot d\vec{a} = \int_{\text{loop}} \left(\frac{\mu_0 N I}{2\pi s}\hat{\varphi}\right)\cdot\left(da\,\hat{\varphi}\right)$$

$$= \frac{\mu_0 N I}{2\pi}\int_a^b\int_0^h \frac{dz\,ds}{s}$$

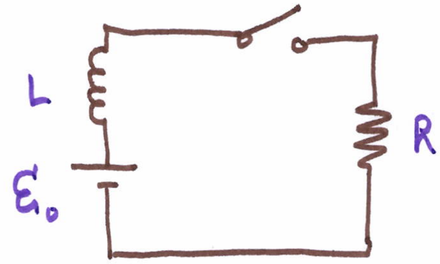
$$= \frac{\mu_0 N I h}{2\pi}\left[\ln(b) - \ln(a)\right]$$

Total flux through toroid is N-times flux through single loop thus,

$$\Phi_{\text{TOTAL}} = N\Phi_1 = \left(\frac{\mu_0 N^2 h \ln(b/a)}{2\pi}\right) I$$

$$\therefore \quad \boxed{L = \frac{\mu_0 N^2 h \ln(b/a)}{2\pi}}$$

$\boxed{E3}$



$$\mathcal{E}_0 - L\frac{dI}{dt} = IR$$

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{\mathcal{E}_0}{L}$$

DEq^{ns}

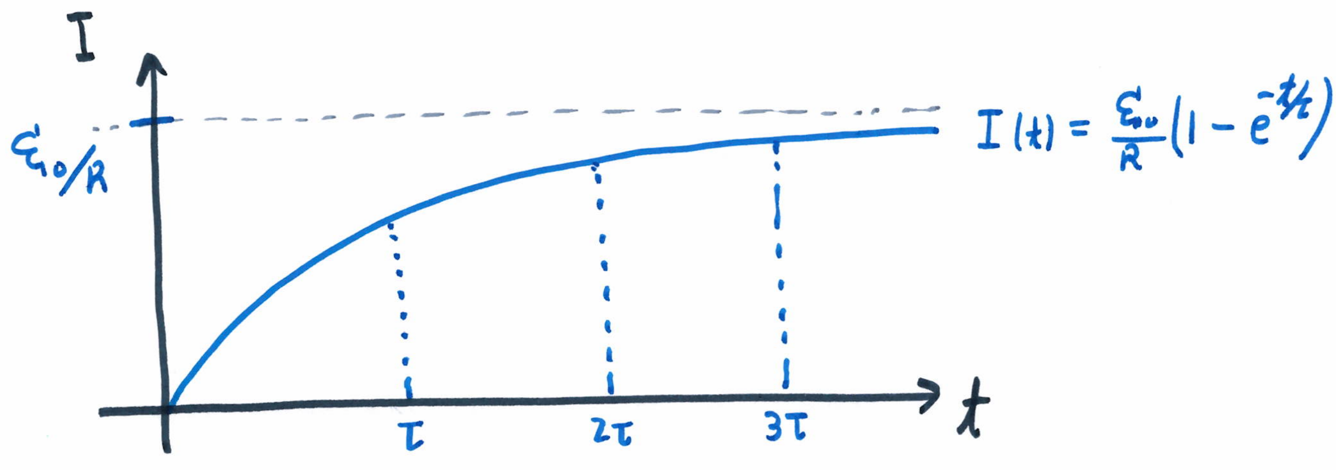
$$I(t) = \frac{\mathcal{E}_0}{R} + c_1 \exp\left(\frac{-Rt}{L}\right)$$

$$I(0) = 0 \implies c_1 = -\mathcal{E}_0/R$$

$$\therefore \boxed{I(t) = \frac{\mathcal{E}_0}{R}\left(1 - \exp\left(-t/\tau\right)\right), \quad \tau = \frac{L}{R}}$$

time constant for RL-circuit.

Remark: inductance resists a change in current, every circuit has inductance.



by 5τ the current on inductor is $\approx 99.3\%$ of the steady state current $\frac{\mathcal{E}_0}{R}$

The time constant $\tau = \frac{L}{R}$ sets the time scale on which the circuit is dynamic.

Energy in Magnetic Fields

Work done per unit time in circuit against the back emf is given by

$$\frac{dW}{dt} = -\mathcal{E}_L = L I \frac{dI}{dt}$$

$$\Rightarrow \boxed{W = \frac{1}{2} L I^2}$$

(assuming $I(0) = 0$, this is energy required to get current to I after some time)

How to understand energy of magnetic field generated from surface or volume currents? Following Griffiths (p 331)

$$\Phi = \int \vec{B} \cdot d\vec{a}$$

$$= \int (\nabla \times \vec{A}) \cdot d\vec{a}$$

$$= \oint \vec{A} \cdot d\vec{l} = L I \quad \Rightarrow W = \frac{1}{2} I \oint \vec{A} \cdot d\vec{l}$$

$$\Rightarrow W = \frac{1}{2} \oint (\vec{A} \cdot \vec{I})\, dl$$

$$\Rightarrow W = \frac{1}{2} \int_{V} (\vec{A} \cdot \vec{J})\, d\tau$$

~~Do better~~ Sorry. Continuing, since $\nabla \times \vec{B} = \mu_0 \vec{J}$ we can eliminate $\vec{J}$ and find,

$$W = \frac{1}{2\mu_0} \int_{V} \vec{A} \cdot (\nabla \times \vec{B})\, d\tau$$

Recall $\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$

thus $\vec{A} \cdot (\nabla \times \vec{B}) = \vec{B} \cdot \vec{B} - \nabla \cdot (\vec{A} \times \vec{B})$ (since $\nabla \times \vec{A} = \vec{B}$ in our context)

Hence,

$$W = \frac{1}{2\mu_0}\left[\int_{\mathcal{V}} B^2 d\tau - \int_{\mathcal{V}} \nabla \cdot (\vec{A} \times \vec{B})\, d\tau \right]$$

$$W = \frac{1}{2\mu_0}\int_{\mathcal{V}} B^2 d\tau - \oint_{\partial \mathcal{V}} (\vec{A} \times \vec{B}) \cdot d\vec{a}$$

$$\Rightarrow \boxed{W = \frac{1}{2\mu_0}\int_{\mathcal{V}} B^2 d\tau}$$ if we let $\mathcal{V} \to$ all space

Compare to electric field energy,

$$\boxed{\begin{aligned} W_{\text{electric}} &= \frac{1}{2}\int (V\rho)\, d\tau = \frac{\varepsilon_0}{2}\int E^2 d\tau \\ W_{\text{magnetic}} &= \frac{1}{2}\int (\vec{A} \cdot \vec{J})\, d\tau = \frac{1}{2\mu_0}\int B^2 d\tau \end{aligned}}$$

Remark: Example 7.14 is nice, but we're moving on.

MAXWELL'S EQUATIONS : MAXWELL'S CORRECTION TERM

Thus far we have analyzed Maxwell's Equations in the following form:

① $\nabla \cdot \vec{E} = \rho/\epsilon_0$ (GAUSS' LAW)

② $\nabla \cdot \vec{B} = 0$ (NO MAGNETIC MONOPOLES)

③ $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (FARADAY'S LAW)

④ $\nabla \times \vec{B} = \mu_0 \vec{J}$ (AMPERE'S LAW)

Are these mathematically consistent? Notice the fundamental identity of vector calculus that $\nabla \cdot (\nabla \times \vec{F}) = 0$ will give us pause, ② and ③,

$$\nabla \cdot (\nabla \times \vec{E}) = \nabla \cdot \left(-\frac{\partial \vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla \cdot \vec{B}) = 0$$

No trouble yet, let's move on to ① and ④,

$$\nabla \cdot (\nabla \times \vec{B}) = \nabla \cdot (\mu_0 \vec{J}) = \underbrace{\mu_0 \nabla \cdot \vec{J}}$$

in magnetostatics we assume $\nabla \cdot \vec{J} = 0$ for steady currents and so

$$\nabla \cdot (\nabla \times \vec{B}) = 0$$

$\underbrace{\text{But, what if } \nabla \cdot \vec{J} \neq 0\text{?}}$

Problematic, something must be modified.

Maxwell proposed Ampere's Law be modified,

$$\boxed{\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}}$$

Let's see how this repairs the problem,

$$\begin{aligned}
\nabla \cdot (\nabla \times \vec{B}) &= \nabla \cdot \left(\mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \\
&= \mu_0 \left(\nabla \cdot \vec{J} + \varepsilon_0 \nabla \cdot \frac{\partial \vec{E}}{\partial t} \right) \\
&= \mu_0 \left(\nabla \cdot \vec{J} + \frac{\partial}{\partial t} \left(\varepsilon_0 \nabla \cdot \vec{E} \right) \right) \\
&= \mu_0 \left(\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} \right) \\
&= 0
\end{aligned}$$

Gauss' Law

conservation of charge.