LECTURE 25: DEFINITION OF MULTIVARIATE INTEGRAL & DOUBLE INTEGRAL CALCULATION...

pgs. 265 – 285 in my 2020 notes.

Chapter 6

integration

In this chapter we study integration of functions of two, three or more variables. The integral is a continuous summation. In particular, we can begin with a finite summation which approximates some quantity. However, as the quantity we consider such as area, volume, mass etc. depends on several variables we have to use a sum which covers some area or volume which describes the possible values of the variables. It is convenient to write such a sum in terms of a mesh which labels each approximating region in terms of the variables. For example, if there are two variables the the approximation is naturally written as a double sum $\sum_i \sum_j$ whereas if there are three variables then we face $\sum_i \sum_j \sum_k$. If we allow the number of approximating areas, or volumes, etc. to shrink to zero as we take the number of such approximating objects to infinity then this brings us to the integral. This is in direct analogy with our development of the single-variate integral from the Riemann sum.

Let me briefly describe the structure of this chapter. In the first section I give the definitions in terms of multiple summations and we detail the properties of multiple integrals. Significantly, we also cover Fubini's wonderful theorems which allow us to calculate multiple integrals by simple iteration of ordinary single-variate integrals. The first section concludes by examining a variety of applications to area, volume and net-quantity as seen from integrals over various area or volume densities. In Section 2 we study double integrals over TYPE I and II Cartesian domains. In Section 3 we study triple integrals with Cartesian coordinates. Section 4 is largely qualitative, we seek to describe the motivation for the change of variables theorem for multiple integrals. Then in Sections 5 and 6 we study the analog of u-substitution for double and triple integrals. Coordinate change is an important tool going forward as the choice of the right coordinate system can sometimes reduce the computational complexity of a problem by a great measure. Finally, in Section 8 we introduce a direct geometric method to understand the structure of dA or dV in non-Cartesian coordinates. In addition, a novel construction called the **wedge product** is introduced and we see how it recovers the calculations of determinants in a simple algebraic fashion.

6.1 definition and interpretations of integration

We begin by defining the double and triple integrals over rectangular domains as the natural extension of the single-variate Riemann sum. Recall,

$$\int_a^b f(x)\,dx \equiv \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \qquad : \qquad \Delta x = \frac{b-a}{n}$$

In other words, the integral is an $f(x)$-weighted sum. Of course, this represents the signed-area. But, the essence of the formula is that the integral is a continuous summation. In view of this observation, the following definitions are natural:

Definition 6.1.1.

integrals are defined as the limit of a weighted sum of f

$$\iint_R f(x,y)\,dA \equiv \lim_{n\to\infty}\lim_{k\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{k} f(x_i^*, y_j^*)\Delta x\Delta y$$

dA = dx dy

$$R = [a,b]\times[c,d] \text{ and } \Delta x = \frac{b-a}{n} \text{ while } \Delta y = \frac{d-c}{k}$$

$$\iiint_B f(x,y,z)\,dV \equiv \lim_{m\to\infty}\lim_{n\to\infty}\lim_{k\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{k} f(x_i^*, y_j^*, z_l^*)\Delta x\Delta y\Delta z$$

dV = dx dy dz

$$B = [a,b]\times[c,d]\times[p,q] \text{ and } \Delta x = \frac{b-a}{m}, \Delta y = \frac{b-c}{n}, \Delta z = \frac{q-p}{k}$$

We note that in Cartesian coordinates $dA = dx\,dy$ = infinitesimal area element in xy-plane. $dV = dx\,dy\,dz$ = infinitesimal volume element. As in calculus I and II, the sample points are chosen randomly and the method in which they are chosen is washed away in the limit. In practice the limit is rarely seen, instead the F.T.C or evaluation rule and here the Fubini theorem will keep us from ever using the definition directly[1]

Let me expand on the specialization of the definition offered above. I have stated the definition in terms of rectangular regions, but in general we might like to calculate integrals over more general regions. For example, we might like to integrate $f(x,y)$ over a disk, or $f(x,y,z)$ over an ellipsoid. I will not attempt to write the multiple Riemann sum for an integral over such a region, however, I will make some unjustified claims which relate the rectangular region integrals to the more general type. The basic idea is this: if $S = S_1 \cup S_2$ then $\int_S f = \int_{S_1} f + \int_{S_2} f$ where I intend this notation to include integrals over areas, volumes and even n-volumes for $n > 3$. In words, the integral over some region is given by adding the integrals over subregions whose union forms the total region. It is geometrically evident that a general region can always be written as a union of rectangular regions. This is not too hard to see in $n = 2, 3$, however, it is also clear the union may need to be over an infinity of rectangular regions. If we suppose the integration region is closed and bounded[2] then analysis beyond this course verifies what we have already *seen*. We can arrange weighted sums over non-rectangular regions as sums of rectangular regions. Furthermore, as we refine the partition

[1] THANKFULLY!

[2] this makes the region compact

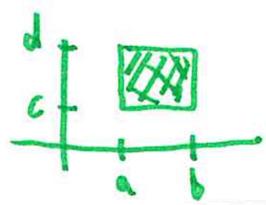
of the region into finer and finer subregions the approximate sums will coverge[3] to a unique value which we call the **integral**. Fortunately, the subtlety I describe here has little to do with our aims in this chapter. The properties and theorems which we soon discover allow us to trade the direct computation of multiple Riemann sums for simple algebraic maniupulation. Moreover, when those methods fail, in this modern age, we may rely on numerical techniques for problems which defy algebraic methods.

Several properties of the integral follow directly from the properties of the limit itself: let R be a closed and bounded region in what follows below:

Proposition 6.1.2.

$$\iint_R [f(x,y)+g(x,y)]dA = \iint_R f(x,y)dA + \iint_R g(x,y)dA$$
$$\iint_R cf(x,y)dA = c\iint_R f(x,y)dA$$
$$f(x,y) \geq g(x,y)\ \forall (x,y) \in R \quad \Rightarrow \quad \iint_R f(x,y)dA \geq \iint_R g(x,y)dA$$

Likewise for $f(x,y,z)$ and $g(x,y,z)$ over a closed and bounded solid region. We assume that f, g are continuous *almost everywhere*[4]. Meaning we can integrate $f(x,y)$ if it has a finite number of curve discontinuities, or $f(x,y,z)$ if it has a family number of planar discontinuities. We just chop the integral into a finite number of regions on which f is continuous. The theorem below was known to Cauchy for continuous f in early 19th century.



$[a,b] \times [c,d]$

Theorem 6.1.3. *Fubini's Theorem(weak form) :*

Let $R = [a,b] \times [c,d]$ and let f be a mostly continuous function of $f(x,y)$

$$\iint_R f(x,y)dA = \int_c^d \left(\int_a^b f(x,y)dx \right) dy = \int_c^d \left(\int_a^b f(x,y)dy \right) dx$$

where the expression on the RHS are "iterated integrals" which you work inside out, treating the outside variable as a constant to begin.

Example 6.1.4. *Let $f(x,y) = \sin(x) + y$ and $R = [0,\pi] \times [0,2]$ which means $(x,y) \in R$ iff $0 \leq x \leq \pi$ and $0 \leq y \leq 2$. Integrate f over R.*

[3]here is where the analysis is needed both here and in calculus I where we were also vague on this point if you do some soul searching. Measure theory makes this process careful and general.

[4]one should study measure theory where this is given a precise meaning, we leave that to a later course in analysis

$R = [0,\pi] \times [0,2]$

$$
\begin{aligned}
\iint_R f(x,y)\,dA &= \int_0^\pi \left(\int_0^2 [\sin(x) + y]dy \right) dx && : (\ldots) \text{ added to emphasize order of } \int \\
&= \int_0^\pi [y\sin(x) + \tfrac{1}{2}y^2] \Big|_{0=y}^{2=y} dx && : \text{note } \sin(x) \text{ is a constant w.r.t. the } dy \text{ integration} \\
&= \int_0^\pi [2\sin(x) + 2]dx \\
&= -2\cos(x)|_0^\pi + 2x|_0^\pi \\
&= -2\cos\pi + 2\cos(0) + 2\pi \\
&= \boxed{4 + 2\pi}
\end{aligned}
$$

A good exercise for the reader: calculate $\int_0^2 \int_0^\pi (\sin(x)+y)dx\,dy$ and compare with the result above.

Example 6.1.5. *Let $R = \{(x,y) \mid 0 \le x \le \frac{\pi}{2},\ 0 \le y \le 2\}$. Consider the integral of $y\cos(xy)$ over R.*

$$
\begin{aligned}
\iint_R y\cos(xy)\,dA &= \int_0^2 \left(\int_0^{\pi/2} y\cos(xy)dx \right) dy && : \int \cos(ax)dx = \frac{1}{a}\sin(ax) + c \\
&= \int_0^2 \left(\sin(xy) \Big|_0^{\pi/2} \right) dy \\
&= \int_0^2 \left(\sin\left(\frac{\pi y}{2}\right) - \sin(0) \right) dy = \int_0^2 \sin\left(\frac{\pi y}{2}\right) dy \\
&= \frac{-2}{\pi}\cos\left(\frac{\pi y}{2}\right)\Big|_0^2 \\
&= \frac{-2}{\pi}\Big(\cos(\pi) - \cos(0)\Big) \\
&= \boxed{\frac{4}{\pi}}
\end{aligned}
$$

Remark 6.1.6.

Notice, if we had integrated first over dy then dx in the preceding example then the calculation would have required integration by parts in the dy integration. The point? Swapping the order of the integration can change the difficulty of the calculation.

6.1.1 interpretations of integrals

In this section thus far we have gained some basic experience in **how** to calculate a multivariate integral. In this subsection we turn to the question of **what** the integral represents. The answer for a given integral is far from unique. We must be prepared to think of the integral as a multifaceted tool which solves more than one problem. Geometrically, integration finds signed areas, volumes and more generally n-volumes which defy direct visualization. Physically, integration of density with respect to some quantity over some space yields the total amount of the quantity in the space. I used the term *space* to be deliberately vague, it might be one, two, or three or even n-dimensional.

Let us begin with the basic interpretation:

Geometry: the double integral of f(x,y) over R is the volume of the solid bounded by $z = f(x, y)$ and $z = 0$ for $(x, y) \in R$. The theorem of Fubini can be seen as merely saying you can slice up the volume along x or y crossections. Infinitesimally

$$dV = \underbrace{(Z_{top} - Z_{base})}_{\text{height of box}} \underbrace{dx\,dy}_{\text{area of box}}$$

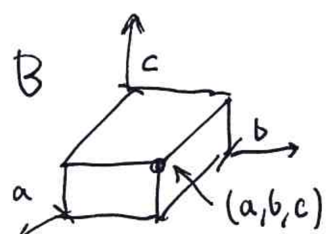
$\iint_R f(x,y)\,dA =$ signed-vol of $z = f(x,y)$ over R

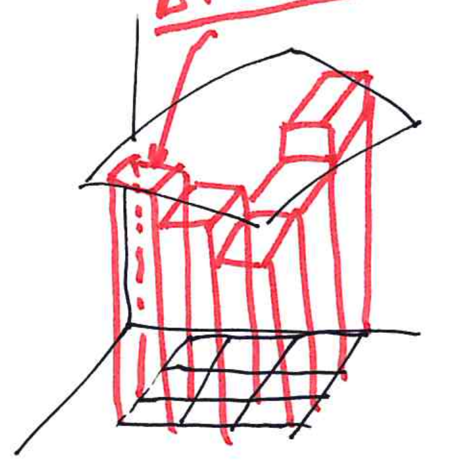
So if $Z_{base} = 0$ and $Z_{top} \geq 0$ then we get the volume, however as in $\int_a^b f(x)dx$ we count volume below the xy-plane as negative so the integral calculates the "signed" volume.

$\Delta V = f(x,y)\Delta x \Delta y$

Example 6.1.7. *Let* $B = \{f(x, y, z) | 0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c\}$

$$\begin{aligned}
\iiint dV &= \int_0^c \int_0^b \int_0^a dx\,dy\,dz \\
&= \int_0^c \int_0^b x\Big|_0^a dy\,dz \\
&= \int_0^c \left(\int_0^b a dy\right) dz \\
&= \int_0^c abdz \\
&= \boxed{abc = V}
\end{aligned}$$





If we integrate 1 over B we find the volume of B. Likewise if we integrate 1 *over a rectangle* $R \subset \mathbb{R}^2$ *we obtain the area.*

Example 6.1.8. *Let* $B = [0, 1] \times [0, 2] \times [0, 3]$*, Let* $\rho = \frac{dm}{dV} = xyz$*. Consider, note, generally the step made in the second equality is only allowed if the integrand f factors into a product of functions of one variable:* $f(x, y, z) = f_1(x)f_2(y)f_3(z)$*,*

$$\begin{aligned}
\iiint_B xyzdV &= \int_0^3 \int_0^2 \int_0^1 xyz\,dx\,dy\,dz \\
&= \int_0^3 zdz \int_0^2 ydy \int_0^1 xdx \\
&= \left(\frac{1}{2}z^2\Big|_0^3\right)\left(\frac{1}{2}y^2\Big|_0^2\right)\left(\frac{1}{2}x^2\Big|_0^1\right) \\
&= \frac{1}{8}(3)^2(2)^2 \\
&= \boxed{\frac{27}{2}}
\end{aligned}$$

$dm = \rho\, dV$
$= xyz\, dV$

What is the meaning of such an integration? Well, if ρ *denotes mass density then* $\rho = dm/dV$ *and it follows* $\rho\, dV = dm$*. Therefore, an object occupying the space B with density* $\rho = xyz$ *has mass* m *as calculated below:*

$$m = \int_B dm = \iiint_B \rho dV = \frac{27}{2}.$$

Or you could interpret it as $\rho = dq/dV$ where dq is the tiny bit of charge in the tiny volume dV hence the total charge in B is $q = \int \rho dV = 27/2$. I'm sure you could imagine other densities.

Example 6.1.9. *Another interpretataion of $\iint_R f(x,y)dA$is that $f(x,y)$ represents an area density. So say $f(x,y) = \sigma(x,y)$*

$$\sigma(x,y) = \frac{dq}{dA} \quad \Rightarrow \quad q = \iint_R \sigma(x,y)dA = \text{ charge on the planar region } R.$$
$$\sigma(x,y) = \frac{dm}{dA} \quad \Rightarrow \quad m = \iint_R \sigma(x,y)dA = \text{ mass of the rectangle } R.$$

Students often insist that $\int_a^b f(x)\,dx$ represents a signed-area. This is true, but, it is not the only interpretation. Consider:

Example 6.1.10. *Another interpretation of $\int_a^b f(x)dx$ is that $f(x)$ reoresents a linear density. So say $f(x) = \lambda(x)$ and,*

$$\lambda(x) = \frac{dq}{dx} \quad \Rightarrow \quad q = \int_a^b \lambda(x)dx \qquad \& \qquad \lambda = \frac{dm}{dx} \Rightarrow m = \int_a^b \lambda(x)dx.$$

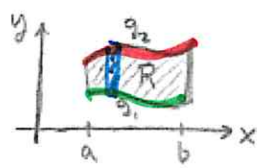
Remark 6.1.11.

Linear density is more exciting once we know about line-integrals. At the moment, we just have the technology to calculate the total amount of some substance whose density is given along a line-segment. The integral with respect to arclength we discuss later will allow us to generalize such calculation to curves.

6.2 Double Integration over General Regions

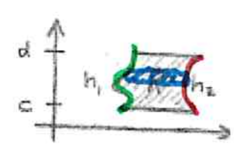
Given an arbitrary connected region in the xy-plane there are two primary descriptions of the region, say R (not necessarily a rectangle anymore). We **define** TYPE I and II as follows:

TYPE I: $\begin{cases} a \leq x \leq b \\ g_1(x) \leq y \leq g_2(x) \end{cases}$



$dA = (g_2 - g_1)dx$

TYPE II: $\begin{cases} c \leq y \leq d \\ h_1(y) \leq x \leq h_2(y) \end{cases}$



$dA = (h_2 - h_1)dy$

Of course, you can imagine region which don't conveniently fit either TYPE. On the other hand a rectange is both TYPE I and II at once. Observe, for a rectangle $R = [a,b] \times [c,d]$ we have $g_1(x) = c, g_2(x) = d$ to show R is TYPE I and we set $h_1(y) = a, h_2(y) = b$ to show R is TYPE II.

Theorem 6.2.1. *(Fubini, Strong version): Suppose f is mostly continuous.*

Given $R_I = \{(x,y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$a TYPE I region,

$$\iint_{R_I} f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

Given $R_{II} = \{(x,y) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$a TYPE II region,

$$\iint_{R_{II}} f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$

Geometric Proof: A justification of the Theorem above is given by the following geometric argument. Suppose R is TYPE I with $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$ for each $x \in [a,b]$. Furthermore, suppose $f(x,y) \geq 0$ for $(x,y) \in R$. In such a case, $\iint_R f(x,y)dA$ represents the volume bounded by $z = 0$, $z = f(x,y)$ and the cylinder $\partial R \times \mathbb{R}$. If we slice such a shape by cross-sections which are parallel to the yz plane then we may form the volume as the union of slices each with thickness dx. In particular, at fixed $x = x_o$ we obtain $dV = A(x_o)dx$ where $A(x_o)$ is the area of the slice of the volume by $x = x_o$. Observe, $g_1(x_o) \leq y \leq g_2(x_o)$ and $dA = zdy = f(x_o, y)dy$ thus

$$A(x_o) = \int_{g_1(x_o)}^{g_2(x_o)} f(x_o, y)dy$$

Now, replace x_o with x and note to find the net-volume we simply sum the volumes of each slice which amounts to integrating $dV = A(x)dx$ from $x = a$ to $x = b$:

$$V = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)dy \right)$$

But, the volume V is also given by $\iint_R f(x,y)dA$ therefore we find the theorem true for TYPE I regions where $f(x,y) \geq 0$. A similar argument supports the theorem for TYPE II regions.

Of course, we also must consider functions which take negative values. We extend our argument thus far to functions which take negative values. We just chop the given region into smaller TYPE I and II regions on which f is nonzero on the interior of each region. Then apply the argument we already offered to the positive value regions and likewise apply the same argument to $-f$ on the subregions on which $f < 0$ hence $-f > 0$ and we may yet again recycle the argument above. Finally, sum the integrals on each subregion to obtain the desired result. Technically, there could be infinitely many regions on which f is negative so we omit a nontrivial analysis here. Indeed, this assertion that the calculation by cross-sections must yield the same value also hides a nontrivial analysis. Anytime we make some argument involving a rearrangement of infinitely many things we should pause to check our assertions. Unfortunately, the refined, technically correct analysis is beyond this course. Indeed, the argument I present here you'll find in many calculus texts.

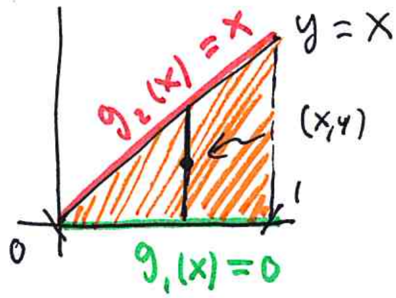
Example 6.2.2. *Let* $R = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq x\}$

$$\iint_R e^{x^2} dA = \int_0^1 \int_0^x e^{x^2}\, dy\, dx$$
$$= \int_0^1 (e^{x^2} y|_0^x) dx$$
$$= \int_0^1 x e^{x^2} dx$$
$$= \frac{1}{2} e^{x^2}\Big|_0^1$$
$$= \frac{1}{2}(e^1 - e^0)$$
$$= \boxed{\frac{1}{2}(e-1)}$$

$0 \leq y \leq 1$

$x_L = y \leq x \leq 1 = x_R$

$$\iint_R e^{x^2} dA = \underbrace{\int_0^1 \left(\int_y^1 e^{x^2} dx \right) dy}_{\text{TYPE II set-up}}$$



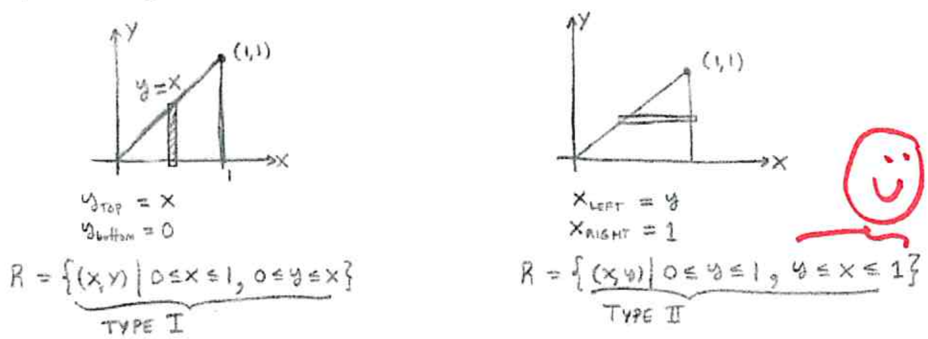
$0 \leq y \leq x$

$0 \leq x \leq 1$

Remark 6.2.3.

We could just as well describe R as a TYPE II region. However, then we'd be forced with $\int e^{x^2} dx$. This is not an elementary integral.

Example 6.2.4. *Using* R *from Example 6.2.2 calculate* $\iint_R e^{y^2} dA$. *Since treating* R *as TYPE I leads us to* $\int e^{y^2} dy$ *we need to make* dx *appear first in the integration. Thus, convert* R *to a TYPE II region. A picture helps:*



Thus, note the second equality below follows as the integral of the constant e^{y^2} *is the product of the integration region length* $(1-y)$ *and the constant,*

$$\iint e^{y^2} dA = \int_0^1 \int_1^y e^{y^2} dx\, dy$$

← oops! $y \leq x \leq 1$ for $0 \leq y \leq 1$

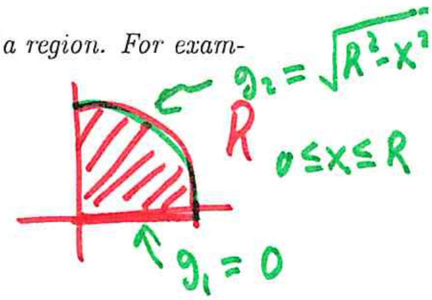
$$= \int_0^1 (1-y) e^{y^2} dy$$
$$= \int_0^1 e^{y^2} dy - \int_0^1 y e^{y^2} dy$$
$$= \int_0^1 e^{y^2} dy - \frac{1}{2}(e-1) \qquad : \textit{curses, I had hoped for better}$$
$$\approxeq 1.463 - \frac{1}{2}(e-1)$$

In the last step, $\int_0^1 e^{y^2}dy$ *required a numerical integration. Sometimes, even swapping the order of the bounds does not make the integral accessible to elementary integration techniques.*

Remark 6.2.5.

Not all integrals result in pretty sums and products, if we just make up some examples on a hunch then it can get ugly. Incidentally while indefinites integrals of e^{x^2} are not known in terms of elementary functions, there are improper integrals of e^{-x^2} which do come out quite nicely. See Ex ?? . We need a few ways to make it easier.

Example 6.2.6. *Another application of double integrals is finding the area of a region. For example,* $S = \{(x,y) \mid 0 \le x \le R,\ 0 \le y \le \sqrt{R^2 - x^2}\ \}$



$$A(R) = \iint_S dA = \int_0^R \int_0^{\sqrt{R^2-x^2}} dydx$$
$$= \int_0^R \sqrt{R^2 - x^2}dx$$

The integration technique to tackle integrals which contain a squareroot is known as trigonometric substitution[5]*. We set* $x = R\cos\theta$ *so* $dx = -R\sin\theta d\theta$ *and* $\sqrt{R^2 - x^2} = R\sin\theta$*. We also must change the bounds. In particular,* $x = R \to \theta = 0$ *and* $x = 0 \to \theta = \pi/2$*. Thus,*

$$\begin{aligned} A(R) &= \int_{\pi/2}^0 -R^2\sin^2\theta d\theta \\ &= R^2\int_0^{\pi/2}\frac{1}{2}(1-\cos 2\theta)d\theta \qquad \text{:we know } \sin^2\theta = \frac{1}{2}(1-\cos 2\theta) \\ &= \frac{R^2}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/2} \\ &= \frac{R^2}{2}(\frac{\pi}{2}) \\ &= \boxed{\frac{\pi R^2}{4}} \end{aligned}$$

If you realized S is a quarter-circle then you should have expected this result.

Remark 6.2.7.

It would be a good exercise to rework this example using polar coordinates. We learn how to change variables in multiple integrals towards the end of this chapter.

Example 6.2.8. *We nay also define the average of a function over R as*

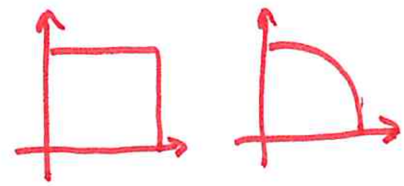
$$\boxed{f_{avg}^R \equiv \frac{1}{A(R)}\iint_R f(x,y)dA}$$

[5] hyperbolic substitution also solves most of the same problems.

Consider $f(x,y) = xy$. If $R = [0,1] \times [0,1]$ and S the quarter circle with R=1 from Example 6.2.6. Do you think $f^R_{avg} > f^S_{avg}$ or vice-versa?

$$\iint_R xy dA = \int_0^1 \int_0^1 xy dx\, dy = \int_0^1 x dx \int_0^1 y dy = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

whereas,



$$\begin{aligned}\iint_S xy dA &= \int_0^1 \int_0^{\sqrt{1-x^2}} xy\, dy\, dx \\ &= \int_0^1 \left(\frac{1}{2} xy^2 \Big|_0^{\sqrt{1-x^2}} \right) dx \\ &= \int_0^1 \frac{1}{2}(x - x^3)\, dx \\ &= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{8}\end{aligned}$$

Thus

$$f^R_{avg} = \frac{1/4}{A(R)} = \frac{1/4}{1} = \frac{1}{4} \qquad whereas \qquad f^{S}_{avg} = \frac{1/8}{A(R)} = \frac{1/8}{\frac{\pi}{4}} = \frac{1}{2\pi}. \quad = f^S_{avg} = \frac{1/8}{A(S)}$$

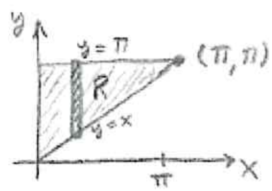
In conclusion, the average of xy is larger on the unit-square since $\frac{1}{4} > \frac{1}{2\pi}$.

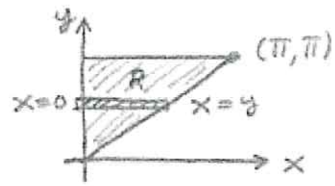
Remark 6.2.9.

Notice $\iint_R xy dA$ was considerably easier than $\iint_S dA$.

Example 6.2.10. *Calculate $\int_0^\pi \int_x^\pi \frac{\sin y}{y}\, dydx$. Notice we need reverse the order of integration to do a TYPE II integration (has $dx\, dy$ instead). Our given integral suggests $0 \le x \le \pi$ and $x \le y \le \pi$.*

$0 \le x \le \pi$
$x \le y \le \pi$





$$\left\{ \begin{array}{c} 0 \le x \le y \\ 0 \le y \le \pi \\ \text{TYPE II} \end{array} \right\}$$

It is graphically clear the region can be recast as type II with $0 \le y \le \pi$ and $0 \le x \le y$. Thus,

$$\iint_R \frac{\sin y}{y} dA = \int_0^\pi \int_0^y \frac{\sin y}{y} dx dy = \int_0^\pi \sin y\, dy = -\cos y \Big|_0^\pi = \boxed{2}.$$

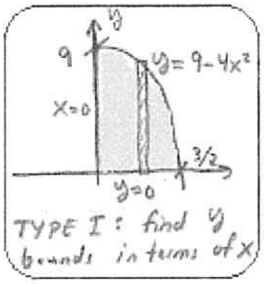
Example 6.2.11. *Calaulate,*

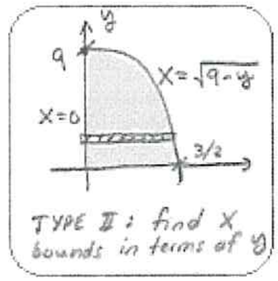
$$\iint_S 16x = \int_0^{3/2} \int_0^{9-4x^2} 16x\, dy\, dx = \int_0^{3/2} 16x(9 - 4x^2) dx \quad \star$$
$$\begin{aligned} &= \int_0^{3/2} (144x - 64x^3) dx \\ &= 72x^2|_0^{3/2} - 16x^4|_0^{3/2} \\ &= 72(3/2)^2 - 16(3/2)^4 \\ &= 162 - 81 = \boxed{81}\end{aligned}$$

At $(\star)$ *we noted that* $16x$ *is a constant w.r.t. the* dy *integration. Hence, we simply multiply by the length of the integration region which is* $9-4x^2$.

Next, let us reverse the order of integration for fun. Consider the graph:

$0 \le y \le 9-4x^2$
$0 \le x \le 3/2$
read from given $\iint$





Note $9-4^2=0 \Rightarrow x^2=\frac{9}{4}, y=9-4x^2$ *is a parabola with x-intercepts* $x \pm 3/2$ *and y-intercept* 9. *Solve* $x^2=\frac{1}{4}(9-y)$ *for* x *and keep positive root:* $x=\frac{1}{2}\sqrt{9-y}$. *Thus,*

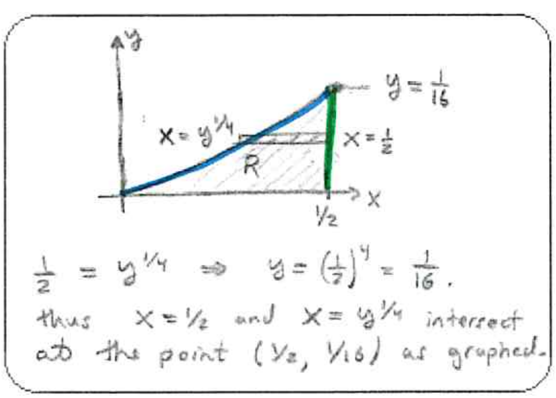
$$\begin{aligned}
\iint_R 16x\,dA &= \int_0^9 \int_0^{\frac{1}{2}\sqrt{9-y}} 16x\,dx\,dy \\
&= \int_0^9 \left(8x^2 \Big|_0^{\frac{1}{2}\sqrt{9-y}} \right) \\
&= \int_0^9 2(9-y)\,dy \\
&= (18y-y^2)\Big|_0^9 \\
&= 18(9)-81 \\
&= \boxed{81}
\end{aligned}$$

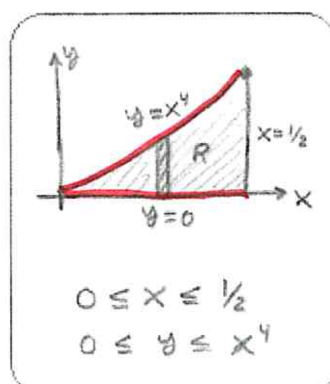
Example 6.2.12. *Calculate*

$$\int_0^{1/16} \int_{y^{1/4}}^{1/2} \cos(16\pi x^5)\,dx\,dy.$$

$y^{1/4} \le x \le 1/2$

Seems changing bounds may be helpful here. To begin $0 \le y \le 1/16$ *and* $y^{1/4} \le x \le \frac{1}{2}$ *which is type II, lets graph to guide our conversion to TYPE I,*





$0 \le y \le x^4$
$0 \le x \le 1/2$

$$\begin{aligned}
\iint_R \cos(16\pi x^5)dA &= \int_0^{1/2}\int_0^{x^4} \cos(16\pi x^5)dydx \\
&= \int_0^{1/2} \cos(16\pi x^5)\left(\int_0^{x^4} dy\right)dx \\
&= \int_0^{1/2} x^4\cos(16\pi x^5)dx \\
&= \frac{1}{80\pi}\sin(16\pi x^5)\Big|_0^{1/2} \\
&= \frac{1}{80\pi}\left(\sin\left(\frac{16\pi}{32}\right) - \sin(0)\right) \\
&= \boxed{\frac{1}{80\pi}.}
\end{aligned}$$

Remark 6.2.13.

The geometric arguments to set-up TYPE I or II should be familiar from your study of areas bounded by curves in single variable calculus. We said TYPE II regions needed horizontal slicing whereas TYPE I were vertically sliced. Notice, for TYPE I: where $y_{base}(x) \leq y \leq y_{top}(x)$ for $a \leq x \leq b$

$$A = \int_a^b \int_{y_{base}(x)}^{y_{top}(x)} dy\,dx = \int_a^b \Big(y_{top}(x) - y_{base}(x)\Big)dx$$

Whereas for TYPE II: where $x_{left}(y) \leq x \leq x_{right}(y)$ for $c \leq y \leq d$

$$A = \int_c^d \int_{x_{left}(y)}^{x_{right}(y)} dx\,dy = \int_c^d (x_{right}(y) - x_{left}(y))dy$$

Thus, in retrospect, we calculated double integrals in disguise in our previous course.

Example 6.2.14.

$$\begin{aligned}
\int_{\pi/6}^{\pi/2}\int_{-1}^{5} \cos(y)dx\,dy &= \int_{\pi/6}^{\pi/2}\left(\cos(y)x\Big|_{-1=x}^{5=x}\right)dy \\
&= \int_{\pi/6}^{\pi/2} 6\cos(y)dy \\
&= 6\sin(y)\Big|_{\pi/6}^{\pi/2} \\
&= 6\sin(\pi/2) - 6\sin(\pi/6) = 6 - 3 = \boxed{3.}
\end{aligned}$$

Example 6.2.15.

$$
\begin{aligned}
\int_0^1 \int_0^3 e^{x+3y}\,dx\,dy &= \int_0^1 \int_0^3 e^x e^{3y}\,dx\,dy \\
&= \int_0^1 e^{3y}\left(\int_0^3 e^x\,dx\right)dy \\
&= \int_0^1 e^{3y}\left(e^x\Big|_{x=0}^{x=3}\right)dy \\
&= \int_0^1 (e^3-1)e^{3y}\,dy \\
&= \left[\frac{e^3-1}{3}e^{3y}\right] e^{3y}\Big|_0^1 \\
&= \left(\frac{e^3-1}{3}(e^3-1)\right) \\
&= \boxed{\frac{1}{3}(e^3-1)^2}
\end{aligned}
$$

$$= \left(\int_0^3 e^x\,dx\right)\left(\int_0^1 e^{3y}\,dy\right)$$

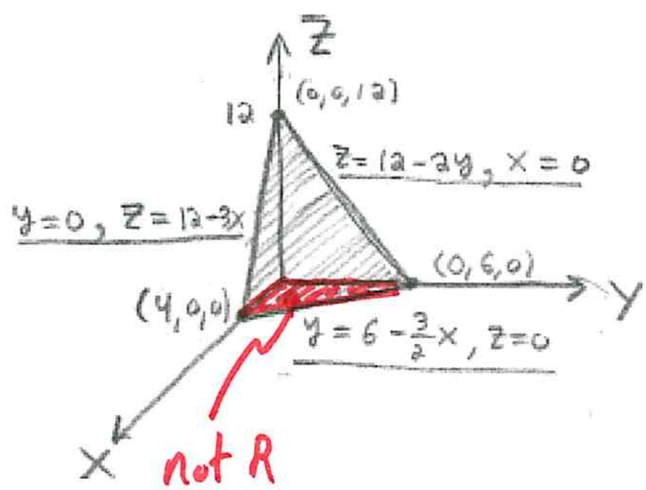
$$= (e^3-1)\left(\frac{1}{3}(e^3-1)\right)$$

Example 6.2.16. *Find volume of the solid under the plane $3x+2y+z=12$ and above the rectangle*

$$R = \{(x,y,0) \mid 0 \leq x \leq 1, -2 \leq y \leq 3\}.$$

$z = 12 - 3x - 2y = f(x,y)$

Solution: *We ought to integrate $z = 12 - 3x - 2y \equiv f(x,y)$ on R. This gives the sum of volumes with height Z. Well, let's be careful, it gives the signed volume hopefully $f(x,y) \geq 0$ for $(x,y) \in R$. Let's pause to verify the geometry is arranged as the problem statement suggests.*



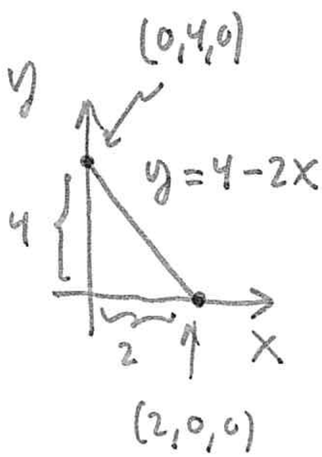
As you can see the graph $z = 12 - 3x - 2y$ is entirely above the xy-plane for the given region. In particular, $0 \leq x \leq 1$ with $-2 \leq y \leq 3$ puts $z = f(x,y) > 0$. Therefore, to find the volume of the *solid we integrate:*

$$
\begin{aligned}
V &= \iint_R f\,dA \\
&= \int_{-2}^{3}\int_0^1 (12 - 3x - 2y)dx\,dy \\
&= \int_{-2}^{3}\left[12x - \frac{3}{2}x^2 - 2yx\right]\Big|_0^1 \\
&= \int_{-2}^{3}\left[12 - \frac{3}{2} - 2y\right]dy \\
&= \int_{-2}^{3}\left(\frac{21}{2} - 2y\right)dy \\
&= \left(\frac{21}{2}y - y^2\right)\Big|_{-2}^{3} \\
&= \frac{105}{2} - (9-4) = \frac{105-10}{2} = \boxed{\frac{95}{2}}.
\end{aligned}
$$

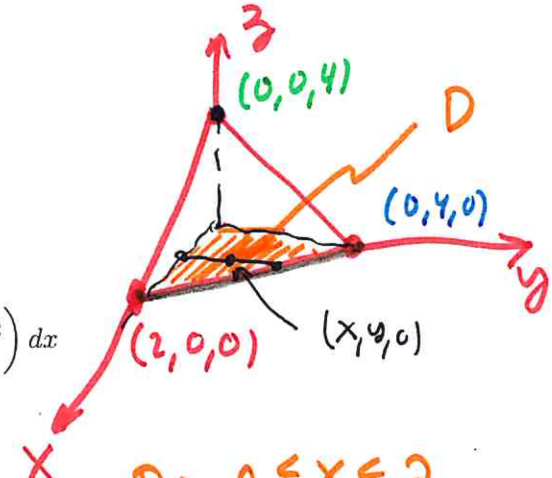
Example 6.2.17. *Find the volume of the tetrahedron enclosed by coordinate planes $x = 0$, $y = 0$, $z = 0$ and the plane $2x + y + z = 4$.*

$z = 4 - 2x - y = f(x,y)$

Solution: *the plane $z = 4 - 2x - y$ intersects the xy-plane along the line given by $z = 0$ and $y = 4 - 2x$. Thus the tetrahedron has $0 \leq z \leq 4 - 2x - y$ and $0 \leq y \leq 4 - 2x$. Finally, when $z = y = 0$ we obtain $0 = 4 - 2x$ hence $x = 2$. It follows $0 \leq x \leq 2$ for the tetrahedron. Therefore, the integration below gives the volume:*



$$
\begin{aligned}
V &= \iint_D (4 - 2x - y)dA \\
&= \int_0^2\int_0^{4-2x} (4 - 2x - y)dy\,dx \\
&= \int_0^2 \left((4-2x)y - \frac{1}{2}y^2\Big|_0^{4-2x}\right)dx \\
&= \int_0^2 \left((4-2x)(4-2x) - \frac{1}{2}(4-2x)^2\right)dx \\
&= \int_0^2 \frac{1}{2}(4-2x)^2dx \\
&= \int_0^2 \frac{1}{2}(16 - 16x + 4x^2)dx \\
&= \int_0^2 (8 - 8x + 2x^2)dx \\
&= \left(8x - 4x^2 + \frac{2}{3}x^3\right)\Big|_0^2 \\
&= \boxed{\frac{16}{3}}
\end{aligned}
$$



$D: 0 \leq x \leq 2$

$0 \leq y \leq 4 - 2x$

Example 6.2.18. *Calculate the double integral below:*

$$\begin{aligned}
\int_1^4 \int_0^2 (x+\sqrt{y})dx\,dy &= \int_1^4 \left(\frac{1}{2}x^2\Big|_0^2 + x\sqrt{y}\Big|_0^2 \right) dy \\
&= \int_1^4 (2+2\sqrt{y})\,dy \\
&= \left(2y + \frac{4y^{3/2}}{3}\right)\Big|_1^4 \\
&= \left[(2(4)+\frac{4}{3}(\sqrt{4})^3\right] - \left[2+\frac{4}{3}\right] \\
&= \boxed{\frac{46}{3}.}
\end{aligned}$$

Example 6.2.19. *Integrate.*

$$\begin{aligned}
\int_1^2 \int_0^1 \frac{1}{(x+y)^2}dxdy &= \int_1^2 \left(\frac{-1}{(x+y)}\Big|_0^1 \right) dy \\
&= \int_1^2 \left(\frac{-1}{y+1} + \frac{1}{y} \right) dy \\
&= (-\ln|y+1| + \ln|y|)\Big|_1^2 \\
&= \ln\left(\frac{|y|}{|y+1|}\right)\Big|_1^2 \\
&= \ln(2/3) - \ln(1/2) \\
&= \boxed{\ln(4/3)}
\end{aligned}$$

Example 6.2.20. *Integrate $\int_0^1 \int_0^1 xy\sqrt{x^2+y^2}\,dy\,dx$. To begin I make a $u = x^2+y^2$ substitution for which $y=0$ gives $u=x^2$ whereas $y=1$ gives $u=x^2+1$. Also, $du = 2ydy$ as we hold x-fixed in the initial integration.*

$$\begin{aligned}
\int_0^1 \int_0^1 xy\sqrt{x^2+y^2}\,dy\,dx &= \int_0^1 \left(\int_{x^2}^{x^2+1} \frac{1}{2}x\sqrt{u}\,du \right) dx \\
&= \int_0^1 \frac{1}{2}x \left(\frac{2}{3}u^{3/2}\Big|_{x^2}^{x^2+1} \right) dx \\
&= \frac{1}{3}\int_0^1 \left(x(x^2+1)^{3/2} - x(x^2)^{3/2} \right) dx \\
&= \frac{1}{3}\int_0^1 x(x^2+1)^{3/2}dx - \frac{1}{3}\int_0^1 x^4 dx \\
&= \frac{1}{6}\int_1^2 (x^2+1)^{3/2} d(x^2+1) - \frac{1}{15} \qquad \star \\
&= \frac{1}{6}\cdot\frac{2}{5}(2^{5/2}-1) - \frac{1}{15} = \boxed{\frac{2}{15}(2\sqrt{2}-1).}
\end{aligned}$$

At $\star$ I made a $w = x^2+1$ substitution.

Remark 6.2.21.

You may find it easier to go off to the side and calculate difficult integrals indefinitely. Otherwise, you do need to change bounds as I have in the example above.

Example 6.2.22. *Let* $R = \{(x, y) \mid 0 \leq x \leq \pi, 0 \leq y \leq \frac{\pi}{2}\}$. *Calculate:*

$$\begin{aligned}
\iint_R \cos(x+2y)dA &= \int_0^{\pi}\left(\int_0^{\pi/2}\cos(x+2y)dy\right)dx \\
&= \int_0^{\pi}\left(\frac{1}{2}\sin(x+2y)\Big|_{0=y}^{\pi/2=y}\right)dx \\
&= \frac{1}{2}\int_0^{\pi}(\sin(x+\pi)-\sin(x))dx \\
&= \frac{1}{2}\left(-\cos(x+\pi)+\cos(x)\Big|_0^{\pi}\right) \\
&= \frac{1}{2}\Big[(-\cos(2\pi)+\cos(\pi)) - (-\cos(\pi)+\cos(0))\Big] \\
&= \boxed{-2.}
\end{aligned}$$

Example 6.2.23. *Integrate,*

$$\begin{aligned}
\int_0^1\int_y^{e^y}\sqrt{x}\,dx\,dy &= \int_0^1\left(\frac{2}{3}x^{3/2}\Big|_y^{e^y}\right)dy \\
&= \frac{2}{3}\int_0^1(e^{3y/2}-y^{3/2})dy \\
&= \frac{2}{3}\left[\frac{2}{3}e^{3y/2}-\frac{2}{5}y^{5/2}\right]\Big|_0^1 \\
&= \frac{4}{3}\left[\left(\frac{1}{3}e^{3/2}-\frac{1}{5}\right)-\left(\frac{2}{3}-0\right)\right] \\
&= \frac{4}{3}\left(\frac{1}{3}e^{3/2}-\frac{13}{15}\right) \\
&= \boxed{\frac{4}{9}e^{3/2}-\frac{13}{45}}
\end{aligned}$$

Example 6.2.24.

$$\int_0^1\int_0^y e^{y^2}dxdy = \int_0^1\left(xe^{y^2}\Big|_0^y\right)dy = \int_0^1 ye^{y^2}dy = \frac{1}{2}e^{y^2}\Big|_0^1 = \boxed{\frac{1}{2}(e-1)}$$

If we had tried to integrate with respect to y first we would have been stuck since $\int e^{y^2}dy$ *is* <u>*not*</u> *an elementary integral. Sometimes reversing the order of integration makes the problem easier.*

Example 6.2.25. $R = \{(x,y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$. *Calculate,*

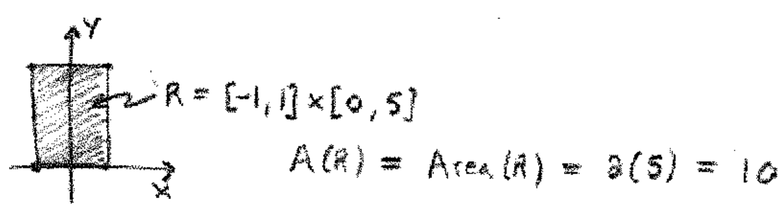
$$\begin{aligned}
\iint_R \frac{1+x^2}{1+y^2} dA &= \int_0^1 \int_0^1 \frac{1+x^2}{1+y^2} dy dx \\
&= \int_0^1 (1+x^2) dx \int_0^1 \frac{1}{1+y^2} dy \\
&= \left(x + \frac{1}{3}x^3 \Big|_0^1 \right) \left(\tan^{-1}(y) \Big|_0^1 \right) \\
&= \frac{4}{3} \left(\tan^{-1}(1) - \tan^{-1}(0) \right) \\
&= \boxed{\frac{\pi}{3}.}
\end{aligned}$$

Example 6.2.26. *Find volume of solid bounded by a top surface of $z = 1 - x^2/4 - y^2/9$ and a base surface given by $R = [-1,1] \times [-2,2]$ on $z = 0$. This makes the sides at $x = -1, x = 1, y = 2, y = -2$. We view R as a subset of xy-plane; that is, we identify R and $R \times \{0\}$ geometrically. In this special case the volune is found by integrating $z = 1 - x^2/4 + y^2/9$,*

$$\begin{aligned}
V = \iint_R \left(1 - \frac{1}{4}x^2 - \frac{1}{9}y^2\right) dA &= \int_{-1}^1 \int_{-2}^2 \left(1 - \frac{1}{4}x^2 - \frac{1}{9}y^2\right) dy\, dx \\
&= 4 \int_0^1 \int_0^2 \left(1 - \frac{1}{4}x^2 - \frac{1}{9}y^2\right) dy\, dx \qquad \star \\
&= 4 \int_0^1 \left(y(1 - \frac{1}{4}x^2) - \frac{1}{27}y^3 \right) \Big|_0^2 dx \\
&= 4 \int_0^1 \left(2 - \frac{1}{2}x^2 - \frac{8}{27} \right) dx \\
&= 4 \left(2 - \frac{1}{6} - \frac{8}{27} \right) \\
&= \boxed{\frac{166}{27}}
\end{aligned}$$

At $\star$ I took advantage of the nature of R and the fact that the integrand was even in both x and y.

Example 6.2.27. *The average of $f(x,y) = x^2 y$ over some region R is defined to be the $\iint_R f(x,y) dA$ divided by the area of $R = \iint_R dA = A(R)$. Let R be region with vertices $(-1,0), (-1,5), (1,5), (1,0)$.*



Thus, we calculate:

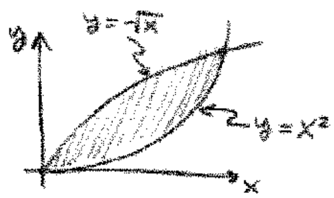
$$f_{avg} = \frac{1}{10} \int_0^5 \int_{-1}^1 x^2 y dx dy = \frac{1}{10} \int_0^5 \frac{1}{3}x^3 y \Big|_{-1}^1 dy = \frac{1}{10} \int_0^5 \frac{2}{3} y dy = \frac{2}{30} \frac{y^2}{2} \Big|_0^5 = \frac{25}{30}.$$

Hence, $\boxed{f_{avg} = 5/6.}$

Example 6.2.28. *Let $D = \{(x, y) \mid$ bounded by $y = \sqrt{x}$ and $y = x^2\}$. Calculate $\iint_D (x + y)dA$*

Solution: *We study D by graphing paired with algebra:*

Let $D = \{(x,y) \mid$ bounded by $y = \sqrt{x}$ and $y = x^2\}$



points of intersection have

$\sqrt{x} = x^2$

$x = x^4$

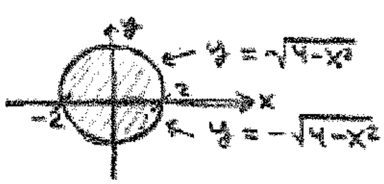
$x(x^3 - 1) = 0 \Rightarrow \underline{x = 0 \text{ or } x = 1}$

Thus the region $D = \{(x, y) | 0 \le x \le 1, x^2 \le y \le \sqrt{x}\}$. Now we can calculate an integral over the integration as follows:

$$
\begin{aligned}
\iint_D (x + y)dA &= \int_0^1 \int_{x^2}^{\sqrt{x}} dy\, dx \\
&= \int_0^1 \left(xy + \frac{1}{2}y^2\right)\Big|_{x^2}^{\sqrt{x}} dx \\
&= \int_0^1 \left[x^{3/2} + \frac{1}{2}x - x^3 - \frac{1}{2}x^4\right] dx \\
&= \left(\frac{2}{5}x^{5/2} + \frac{1}{4}x^2 - \frac{1}{4}x^4 - \frac{1}{10}x^5\right)\Big|_0^1 \\
&= \frac{2}{5} + \frac{1}{4} - \frac{1}{4} - \frac{1}{10} \\
&= \boxed{\frac{3}{10}.}
\end{aligned}
$$

Example 6.2.29. *$D = \{(x, y) | x^2 + y^2 \le 4\}$. Calculate $\iint_D (2x - y)dA$.*

Solution: *our first task is to describe D via inequalities for a typical point in D:*



$-2 \le x \le 2$

$-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$

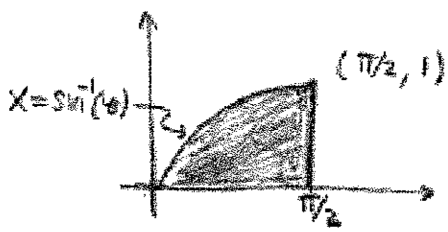
Now we can integrate,

$$
\begin{aligned}
\iint_D (2x-y)dA &= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (2x-y)\,dy\,dx \\
&= \int_{-2}^{2} \left(2xy - \frac{1}{2}y^2\right)\Bigg|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx \\
&= \int_{-2}^{2} \left(4x\sqrt{4-x^2} - \frac{1}{2}(4-x^2) + \frac{1}{2}(4-x^2)\right) dx \\
&= \frac{-4}{3}(4-x^2)^{3/2}\Bigg|_{-2}^{2} \\
&= \frac{-4}{3}(0-0) = \boxed{0.}
\end{aligned}
$$

This result is completely unsurprising if we consider the arrangement of values for x and y over the disk of radius 2. There is a perfect balance between positive and negative values.

Example 6.2.30. *Consider the region R defined by $0 \leq y \leq 1$ and $\sin^{-1}(y) \leq x \leq \pi/2$. Let $f(x,y) = \cos(x)\sqrt{1+\cos^2 x}$ and calculate $\iint_R f\, dA$.*

Solution: *we need to reformulate R since integration with respect to x is not obvious. Note:*



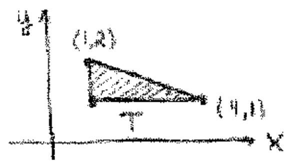
$0 \leq x \leq \pi/2$
$0 \leq y \leq \sin(x)$

Thus

$$
\begin{aligned}
\iint_R f(x,y)\,dA &= \int_0^{\pi/2}\int_0^{\sin(x)} \cos(x)\sqrt{1+\cos^2 x}\,dy\,dx \\
&= \int_0^{\pi/2} \underbrace{\sqrt{1+\cos^2 x}}_{\sqrt{u}}\ \underbrace{\sin(x)\cos(x)\,dx}_{-du/2} \\
&= \frac{-1}{3}(1+\cos^2 x)^{3/2}\Bigg|_0^{\pi/2} \\
&= \frac{-1}{3}(1-2^{3/2}) \\
&= \boxed{\frac{1}{3}(2\sqrt{2}-1)}.
\end{aligned}
$$

Example 6.2.31. *Find volume under $z = xy$ and above the triangle with vertices $(1,1,0),(4,1,0)$ and$(1,2,0)$.*

Solution: *it is natural to identify the given triangle with $T \subset \mathbb{R}^2$ formed by $(1,1)$,$(4,1)$ and $(1,2)$. We picture T below where the top line is $y = \frac{7-x}{3}$ and the bottom line is $y = 1$ for $1 \leq x \leq 4$.*



Notice xy is clearly postive on $T = \{(x,y) \mid 1 \leq x \leq 4,\ 1 \leq y \leq \frac{7-x}{3}\}$ hence the volume is given by:

$$\begin{aligned}
V &= \int_1^4 \int_1^{\frac{1}{3}(7-x)} xy\,dy\,dx \\
&= \int_1^4 \left(\frac{1}{2}xy^2 \Big|_1^{\frac{1}{3}(7-x)} \right) dx \\
&= \int_1^4 \frac{1}{2}x\left(\frac{1}{9}(7-x)^2 - 1\right) dx \\
&= \frac{1}{18}\int_1^4 x\big[(7-x)^2 - 9\big]dx \\
&= \frac{1}{18}\int_1^4 \big[40x - 14x^2 + x^3\big]dx = \boxed{\frac{31}{8}.}
\end{aligned}$$

I'll let you fill in the last few details above. Notice that we can also write T as

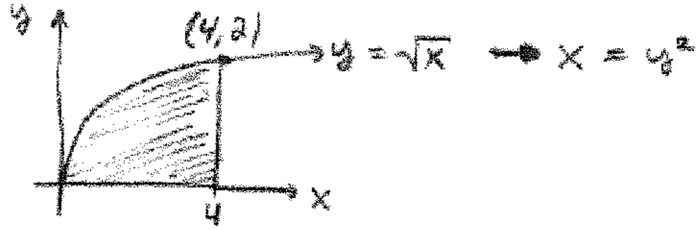
$$T = \{(x,y) \mid 1 \leq y \leq 2, 1 \leq x \leq 7 - 3y\}$$

then we can integrate x first then y.

$$\begin{aligned}
V &= \int_1^2 \int_1^{7-3y} xydxdy \\
&= \int_1^2 \frac{1}{2}yx^2 \Big|_1^{7-3y} dy \\
&= \int_1^2 \frac{1}{2}y\left((7-3y)^2 - 1\right) dy \\
&= \int_1^2 \frac{1}{2}y\left(48 - 42y + 9y^2\right) dy \\
&= \frac{1}{2}\int_1^2 \left(48y - 42y^2 + 9y^3\right) dy \\
&= \frac{1}{2}\left(24y^2 - \frac{42}{3}y^3 + \frac{9}{4}y^4 \Big|_1^2 \right) \\
&= \frac{1}{2}\left(24(4-1) - 14(8-1) + \frac{9}{4}(16-1) \right) \\
&= \frac{1}{2}\left(72 - 98 + \frac{135}{4}\right) = \frac{1}{2}\left(-26 + \frac{135}{4}\right) = \frac{1}{2}\left(\frac{-104+135}{4}\right) = \boxed{\frac{31}{8}.}
\end{aligned}$$

Ok, its messy anyway you slice it.

Example 6.2.32. *Consider the following integral,* $\int_0^4 \int_0^{\sqrt{x}} f(x,y)\,dy\,dx$ *this indicates the integral is over* $0 \leq y \leq \sqrt{x}$ *and* $0 \leq x \leq 4$.



Equivalently we could say $y^2 \leq x \leq 4$ *and* $0 \leq y \leq 2$

$$\int_0^4 \int_0^{\sqrt{x}} f(x,y)\,dy\,dx = \int_0^2 \int_{y^2}^4 f(x,y)\,dx\,dy$$

for problems such as this, you just have to draw the picture and sort it out.

Example 6.2.33. *Let* R *be the region in the* xy*-plane bounded by* $y = 0$, $x = 1$ *and* $y = x^2$. *Calculate* $\iint_R \sqrt{x^3+1}\,dA$.

Solution: *it is clearly unpleasant to integrate with respect to* x *to begin. It follows*[6] *we should view* R *as* $0 \leq y \leq 1$, $\sqrt{y} \leq x \leq 1$. *Hence, integrate:*

$$\begin{aligned}\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}dx\,dy &= \int_0^1 \int_0^{x^2} \sqrt{x^3+1}dydx \\ &= \int_0^1 y\sqrt{x^3+1}\Big|_0^{x^2} dx \\ &= \int_0^1 x^2\sqrt{x^3+1}dx \\ &= \frac{2}{9}\left(x^3+1\right)^{3/2}\Big|_0^1 \\ &= \frac{2}{9}\left(2^{3/2}-1\right) \\ &= \boxed{\frac{2}{9}(2\sqrt{2}-1)}\end{aligned}$$

[6] it may be helpful for you to draw a picture to verify this claim