LECTURE 26: CARTESIAN TRIPLE INTEGRALS

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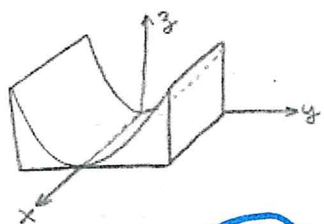
## 6.3 Triple Integration over General Bounded Regions in $\mathbb{R}^3$

Rather than explicitly stating the Fubini Theorem for triple integrals I will simply illustrate with a few examples. Usually we can bound $z$ in terms of $x$ and $y$, then we can bound $y$ in terms of $x$ or vice-versa, that gives two orders of integration. Then other problems allow $x$ to be bound in terms of $y$ and $z$ or possibly $y$ in terms of $x$ and $z$, in total there are six ways to write a particular integral. I don't give general advice on how to rewrite and switch bounds, it's a subtle business and there are far too many cases to enumerate. Generalities aside, let's do a few typical problems.

**Example 6.3.1.** *Let us find the volume of the region between $z = y^2$ and the xy-plane bounded by $x = 0, x = 1, y = 1$ and $y = -1$. Notice $dV = dxdydz$ so integrating $dV$ gives volume $V$,*

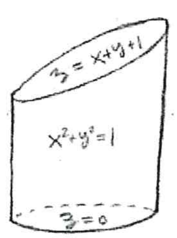



$0 \leq z \leq y^2$
$0 \leq x \leq 1$
$-1 \leq y \leq 1$

- we must integrate w.r.t. z either first or second.

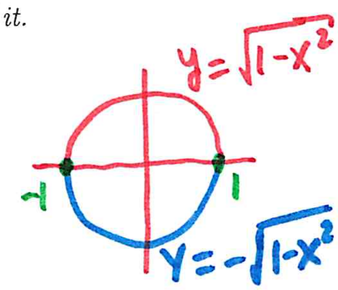
$$
\begin{aligned}
V &= \int_{-1}^{1}\int_{0}^{1}\int_{0}^{y^2} dz\,dx\,dy && \text{: we work inside out as usual} \\
&= \int_{-1}^{1}\int_{0}^{1} y^2\,dx\,dy && \text{: back to 2-d integrals} \\
&= \int_{-1}^{1} y^2\,dy && \text{: back to 1-d integral} \\
&= \frac{1}{3}y^3\Big|_{-1}^{1} \\
&= \frac{1}{3}(1-(-1)^3) \\
&= \boxed{2/3}
\end{aligned}
$$

**Example 6.3.2.** *Consider the cylinder $x^2+y^2 = 1$, let $z = 0$ bound it from below and let $z = x+y+1$ bound it above, call this solid $B$. A sketch of $B$ reveals the inequlities to the right of it.*




$0 \leq z \leq x+y+1$

$0 \leq x^2+y^2 \leq 1 \longrightarrow -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$, $\quad -1 \leq x \leq 1$




*With the geometry settled, we are free to calculate integrals of functions over $B$. For example, we* *can integrate $f(x,y,z) = x$ over $B$ as follows:*

$-1 \leq x \leq 1$
$-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$
$0 \leq z \leq x+y+1$
$(x,y,z) \in B$

$$\iiint_B x\,dV = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{x+1+y} x\,dzdydx$$
$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2+x+xy)dydx$$

*note, the integral of $y$ on the symmetric interval $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$ is zero since the integral of an odd function over a symmetric interval about the origin is zero. Continuing,*

$$\iiint_B x\,dV = \int_{-1}^{1}\left((x^2+x)y\Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\right)dx$$
$$= \int_{-1}^{1} 2x^2\sqrt{1-x^2}\,dx$$
$$= 4\int_0^1 x^2\sqrt{1-x^2}\,dx$$

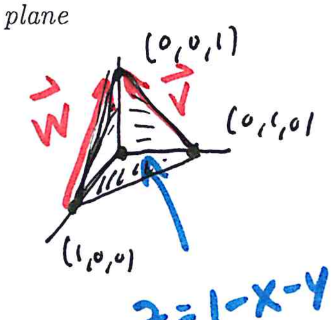
*The last step was justified because the integrand is even in $x$ and the integral of an even function about a symmetric interval of the origin is simply twice the integral of the left or right half of the integral. The remaining intergral can be calculated by a trigonometric subsitution. In particular, let $x = \sin\theta$ thus $dx = \cos\theta\,d\theta$ and $1-x^2 = \cos^2\theta$ and $\sqrt{1-x^2} = \cos\theta$*

$$\int x^2\sqrt{1-x^2}dx = \int \sin^2\theta\cos\theta\cos\theta\,d\theta$$
$$= \int (\sin^2\theta - \sin^4\theta)\,d\theta$$
$$= \int\left(\frac{1}{2}(1-\cos(2\theta)) - \frac{1}{4}(1-2\cos(2\theta)+\cos^2(2\theta))\right)d\theta$$
$$= \int\left(\frac{1}{2} - \frac{1}{2}\cos(2\theta) - \frac{1}{4} + \frac{1}{2}\cos(2\theta) - \frac{1}{4}\cos^2(2\theta)\right)d\theta$$
$$= \int\left(\frac{1}{4} - \frac{1}{8}(1-\cos(4\theta))\right)d\theta$$
$$= \frac{\theta}{8} + \frac{\sin(4\theta)}{32} + C.$$

*Change bounds on $x$ from $x = 0 \to \theta = 0$ and $x = 1 \to \theta = \frac{\pi}{2}$*

$$\iiint_B x dV = 4\left(\frac{\theta}{8} + \frac{1}{32}\sin(4\theta)\right)\Big|_0^{\pi/2} = \boxed{\frac{\pi}{4}}$$

**Example 6.3.3.** *Let $B$ be bounded by coordinate planes and the plane possing through $(0,0,1)$, $(0,1,0)$, and $(1,0,0)$. We find the equation of this plane to begin, note $\vec{v}$ and $\vec{w}$ are on the plane*




$$\vec{v} = (0,0,1) - (0,1,0) = <0,-1,1>$$

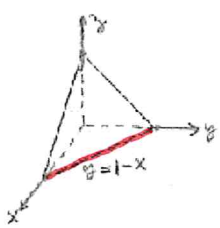
$$\vec{w} = (0,0,1) - (1,0,0) = <-1,0,1>$$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{vmatrix} = \langle -1,-1,-1 \rangle$$

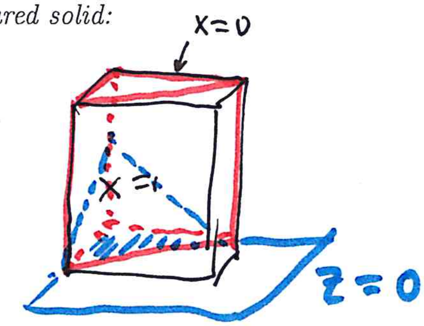
*Use normal $\langle -1,-1,-1 \rangle$ and basepoint $(0,0,1)$ to give plane equation:*

$$-x - y - (z-1) = 0 \quad \Rightarrow \quad z = 1 - x - y$$

*Let's plot it. Note $z = 1 - x - y$ intersects $z = 0$ on the line $y = 1 - x$ in the $xy$-plane and we find the inequalities to bound each coordinate of $(x, y, z)$ found within the pictured solid:*




$$0 \leq z \leq 1 - x - y$$
$$0 \leq y \leq 1 - x$$
$$0 \leq x \leq 1$$




*We calculate the volume of $B$ by integrating $dV$ over $B$:*

$$\begin{aligned}
V = \iiint_B dV &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\, dy\, dx \\
&= \int_0^1 \int_0^{1-x} (1 - x - y) dy\, dx \\
&= \int_0^1 \left( (1-x)y - \frac{1}{2}y^2 \Big|_0^{1-x} \right) dx \\
&= \int_0^1 \left( (1-x)^2 - \frac{1}{2}(1-x)^2 \right) dx \\
&= \int_0^1 \frac{1}{2}(1 - 2x + x^2) dx \\
&= \frac{1}{2}(1 - \frac{2}{2} + \frac{1}{3}) = \boxed{\frac{1}{6}}
\end{aligned}$$

Notice, the bulk of the difficulty is usually in setting-up the integral. The process of calculating an iterated integral is (for most of us) the easy part.

**Example 6.3.4.** *Find the average value of $f(x,y,z) = x$ on the solid region from Example 6.3.3. The average is defined to be.*

$$f^B_{avg} \equiv \frac{1}{vol(B)} \iiint_B f(x,y,z)dV$$

*We just found $vol(B) = 1/6$, let's focus on the $\iiint_B f dV$.*

$$\begin{aligned}
\iint_B f dV &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x\, dz\, dy\, dx \\
&= \int_0^1 x \int_0^{1-x} (1-x-y) dy\, dx \\
&= \int_0^1 \left( (1-x)y - \frac{1}{2}y^2 \Big|_0^{1-x} \right) dx \\
&= \int_0^1 \frac{1}{2} x(1-x)^2 dx \\
&= \int_0^1 \frac{1}{2}(x - 2x^2 + x^3) dx \\
&= \frac{1}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{1}{2}\left(\frac{3}{4} - \frac{2}{3}\right) = \boxed{\frac{1}{24}}.
\end{aligned}$$

*Thus, $f^B_{avg} = \frac{1/24}{1/6} = \frac{1}{4}$. Notice, the result seems reasonable in view of the picture in Example 6.3.3.*

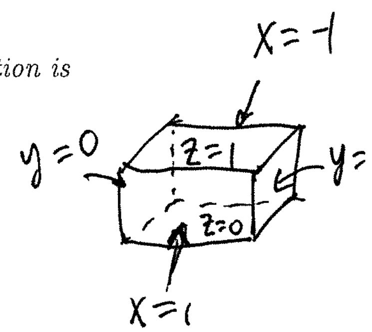
**Example 6.3.5.**

$$\begin{aligned}
\int_0^1 \int_0^z \int_0^{x+z} 6xz\, dy\, dx\, dz &= \int_0^1 \int_0^z \left( 6xyz \Big|_{0=y}^{x+z=y} \right) dx\, dz \\
&= \int_0^1 \int_0^z 6x(x+z)z dx\, dz \\
&= \int_0^1 \int_0^z \left(6x^2 z + 6xz^2\right) dx\, dz \\
&= \int_0^1 \left( 2x^3 z + 3x^2 z^2 \Big|_{x=0}^{x=z} \right) \\
&= \int (2z^4 + 3z^2) dz \\
&= \int_0^1 5z^4 dz \\
&= z^5 \Big|_0^1 \\
&= \boxed{1}
\end{aligned}$$

**Example 6.3.6.** *Let* $E = \{(x,y,z)|0 \le x \le 1, 0 \le y \le x, x \le z \le 2x\}$

$$\begin{aligned}
\iiint_E yz\cos(x^5)dV &= \int_0^1 \left(\int_x^{2x}\left(\int_0^x yz\cos(x^5)dy\right)dz\right)dx \\
&= \int_0^1 \int_x^{2x} \left(z\cos(x^5)\frac{y^2}{2}\bigg|_0^x\right) dz\,dx \\
&= \int_0^1 \int_x^{2x} \left(\frac{1}{2}x^2 z\cos(x^5)\right) dz\,dx \\
&= \int_0^1 \left(\frac{1}{2}x^2\cos(x^5)\frac{z^2}{2}\bigg|_x^{2x}\right) dx \\
&= \int_0^1 \left(\frac{1}{2}x^2\cos(x^5)\frac{1}{2}(4x^2 - x^2)\right) dx \\
&= \int_0^1 \frac{3}{4}x^4\cos(x^5)dx \qquad \boxed{u = x^5, du = 5x^4dx, u(1) = 1, u(0) = 0} \\
&= \int_0^1 \frac{3}{20}\cos(u)du \\
&= \frac{3}{20}(\sin(1) - \sin(0)) = \boxed{\frac{3\sin(1)}{20}}
\end{aligned}$$

**Example 6.3.7.** *Evaluate the integral three different ways. The region of integration is* $E = \{(x,y,z) \mid -1 \le x \le 1, 0 \le y \le 2, 0 \le z \le 1\}$-




$$\begin{aligned}
\iiint_E (xz - y^3)dV &= \int_{-1}^1 \int_0^2 \int_0^1 (xz - y^3)dz\,dy\,dx \\
&= \int_{-1}^1 \int_0^2 \left(\frac{1}{2}x - y^3\right) dy\,dx \\
&= \int_{-1}^1 \left(x - \frac{1}{4}(16)\right) dx = -4(2) = \boxed{-8.}
\end{aligned}$$

*In what follows below,* $xz$ *is an odd-function integrated over symmetric interval about zero vanishes.*

$$\begin{aligned}
\iiint_E (xz - y^3)dV &= \int_0^1 \int_0^2 \int_{-1}^1 (xz - y^3)dx\,dy\,dz \\
&= \int_0^1 \int_0^2 (-xy^3\bigg|_{-1}^1 dydz) \\
&= \int_0^1 \int_0^2 -2y^3 dydz \\
&= \int_0^1 \frac{-2}{4}(2)^4 dz = -8z|_0^1 = \boxed{-8.}
\end{aligned}$$

*There are four other ways to interate the integral. Each will yield* $-8$*. This is Fubini's Theorem for* $\iiint$ *in action.*

**Example 6.3.8.** *The notation on the evaluation bars in this example is optional. You may find it helpful as you begin your study of multivariate integration. You can contrast this notation with the less explicit notationin the example which follows.*

$$
\begin{aligned}
\int_0^3\int_0^1\int_0^{\sqrt{1-z^2}} ze^y\,dx\,dz\,dy &= \int_0^3\int_0^1 ze^y x\Big|_{x=0}^{x=\sqrt{1-z^2}}\,dz\,dy \\
&= \int_0^3\int_0^1 z\sqrt{1-z^2}e^y\,dzdy \\
&= \int_0^3 \left(\frac{-1}{3}(1-z^2)^{3/2}e^y\Big|_{z=0}^{z=1}\right) dy \\
&= \int_0^3 \frac{1}{3}e^y\,dy = \boxed{\frac{1}{3}(e^3-1)}
\end{aligned}
$$

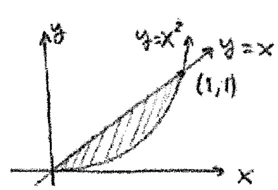
$u = 1 - z^2$

**Example 6.3.9.** *Calculate the integral below:*

$$
\begin{aligned}
\int_0^1\int_0^z\int_0^y ze^{-y^2}\,dx\,dz\,dy &= \int_0^1\int_0^z \left(ze^{-y^2}x\Big|_0^y\right) dy\,dz \\
&= \int_0^1\int_0^z ze^{-y^2}y\,dy\,dz \\
&= \int_0^1 \left(\frac{-1}{2}ze^{-y^2}\Big|_0^z\right) dz \\
&= \int_0^1 \left(\frac{-1}{2}ze^{-z^2}+\frac{1}{2}z\right) dz \\
&= \left(\frac{1}{4}e^{-y^2}+\frac{1}{4}z^2\right)\Big|_0^1 \\
&= \frac{1}{4}\left(e^{-1}+1-1\right) = \boxed{\frac{1}{4e}}.
\end{aligned}
$$

**Example 6.3.10.** *$E$ be the solid region in $\mathbb{R}^3$ bounded by the parabolic cylinder $y = x^2$ and the planes $x = z$, $x = y$ and $z = 0$. Calculate $\iiint_E (x+2y)dV$.*

**Solution:** *we bound $z$ to begin, $0 \le z \le x$. Then a two-dimension picture will do:*




$x^2 \le y \le x$

$0 \le x \le 1$

*Note $(x,y,z) \in E$ implies $0 \leq z \leq x$ for $x^2 \leq y \leq x$ where $0 \leq x \leq 1$. Hence, integrate:*

$$
\begin{aligned}
\iiint_E (x+2y)\,dV &= \int_0^1 \int_{x^2}^{x} \int_0^x (x+2y)\,dz\,dy\,dx \\
&= \int_0^1 \int_{x^2}^{x} (x^2+2yx)\,dy\,dx \\
&= \int_0^1 \left( x^2y + xy^2 \Big|_{x^2}^{x} \right) dx \\
&= \int_0^1 (x^3+x^3-x^4-x^5)\,dx \\
&= \left( \frac{2}{4}x^4 - \frac{1}{5}x^5 - \frac{1}{6}x^6 \right)\Big|_0^1 \\
&= \frac{1}{2} - \frac{1}{5} - \frac{1}{6} = \frac{15-6-5}{30} = \boxed{\frac{2}{15}}
\end{aligned}
$$

**Example 6.3.11.** $E = \{(x,y,z) \mid 0 \leq x \leq 1, 0 \leq y \leq x, x \leq z \leq 2x\}$. *We must integrate with respect to $dz$ then $dx$. That is the natural order here. For example, to integrate $f(x,y,z) = yz\cos(x^5)$ on $E$ we calculate as follows:*

$$
\begin{aligned}
\iiint_E yz\cos(x^5)\,dV &= \int_0^1 \int_0^x \int_x^{2x} yz\cos(x^5)\,dz\,dy\,dx \\
&= \int_0^1 \cos(x^5)\left( \int_0^x \int_x^{2x} yz\,dz\,dy \right) dx \\
&= \int_0^1 \cos(x^5)\left( \int_0^x \frac{1}{2}[(2x)^2 - x^2])y\,dy \right) dx \\
&= \int_0^1 \frac{3}{2}x^2\cos(x^5)\left( \int_0^x y\,dy \right) dx \\
&= \int_0^1 \frac{3}{2}x^2\cos(x^5)\left( \frac{1}{2}y^2 \Big|_0^x \right) dx \\
&= \int_0^1 \frac{3}{4}x^4\cos(x^5) \\
&= \frac{3}{20}\sin(x^5)\Big|_0^1 = \boxed{\frac{3}{20}\sin(1).}
\end{aligned}
$$

Contrast with Ex. 6.3.6

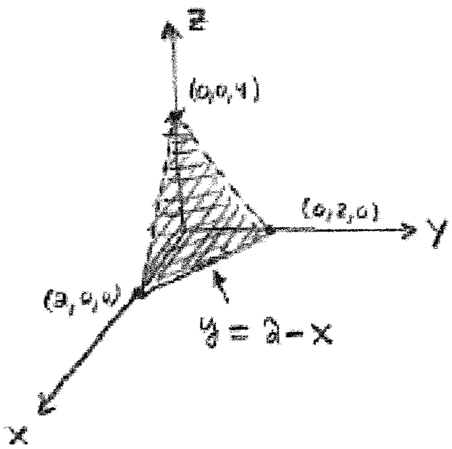
**Example 6.3.12.** *Calculate $\iiint_E y\,dV$ where $E$ is the solid region in the first octant bounded by $x = 0, y = 0, z = 0$ and $2x + 2y + 2z = 4$.*

**(i.)** *$(xz)$-plane $(y = 0) : 2x + z = 4$ yields $z = 4 - 2x$,*

**(ii.)** *$(xy)$-plane $(z = 0) : 2x + 2y = 4$ yields $y = 2 - x$,*

**(iii.)** *$(yz)$-plane $(x = 0) : 2y + z = 4$ yields $z = 4 - 2yt$.*

*these details are not strictly speaking necessary but sometimes it helps to get some additional details to help insure graph is correct.*




So we can describe the region of integration as

$$0 \le z \le 4 - 2x - 2y$$

but what about $x$ & $y$? Note on $(xy)$-plane we have

$$0 \le y \le 2 - x$$

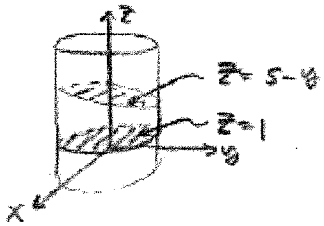
And finally

$$0 \le x \le 2$$

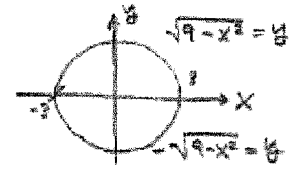
*now integrate,*

$$\begin{aligned}
\iiint_E y\,dV &= \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} y\,dz\,dy\,dx \\
&= \int_0^2 \int_0^{2-x} y(4 - 2x - 2y)\,dy\,dx \\
&= \int_0^2 \int_0^{2-x} \big(2(2-x)y - 2y^2\big)\,dy\,dx \\
&= \int_0^2 \left((2-x)y^2 - \frac{2}{3}y^3\right)\Bigg|_0^{2-x} dx \\
&= \int_0^2 \left((2-x)^3 - \frac{2}{3}(2-x)^3\right) dx \\
&= \int_0^2 \frac{1}{3}(2-x)^3\,dx \\
&= \frac{-1}{12}(2-x)^4\Bigg|_0^2 \\
&= \frac{-1}{12}(0 - 16) = \frac{16}{12} = \boxed{\frac{4}{3}}
\end{aligned}$$

**Example 6.3.13.** *Find volume enclosed by $x^2 + y^2 = 9$ and $y + z = 5, z = 5 - y$ and $z = 1$*




$$1 \le z \le 5-y$$
$$-\sqrt{9-x^2} \le y \le \sqrt{9-x^2}$$
$$-3 \le x \le 3$$




*In view of the diagrams above,*

$$V = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{1}^{5-y} dz\,dy\,dx = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}(4-y)dy\,dx$$
$$= \int_{-3}^{3} 8\sqrt{9-x^2}dx$$

*If we substitute $x = 3\sin\theta$ then $9 - x^2 = 9 - 9\sin^2\theta = 9\cos^2\theta$ and $dx = 3\cos\theta d\theta$. Also $x = 3$ corresponds to $\theta = \pi/2$ whereas $x = -3$ corresponds to $\theta = -\pi/2$. Consequently,*

$$V = \int_{-\pi/2}^{\pi/2} 72\cos^2\theta d\theta$$
$$= 72\int_{-\pi/2}^{\pi/2}\frac{1}{2}(1+\cos 2\theta)d\theta$$
$$= \boxed{36\pi.}$$

**Example 6.3.14.** *Consider $\int_0^1\int_{\sqrt{x}}^1\int_0^{1-y} f(x,y,z)dz\,dy\,dx$. Find five other orders of iterating this integral. Let us consider the cases:*

$$0 \le z \le 1-y \qquad 0 \le z \le 1-y$$
$$\sqrt{x} \le y \le 1 \quad \text{OR} \quad 0 \le x \le y^2$$
$$0 \le x \le 1 \qquad 0 \le y \le 1$$

OR OR

$$0 \le x \le y^2 \qquad 0 \le x \le y^2$$
$$0 \le y \le 1-z \qquad 0 \le z \le 1-y$$
$$0 \le z \le 1 \qquad 0 \le y \le 1$$

OR OR

$$\sqrt{x} \le y \le 1-z \qquad \sqrt{x} \le y \le 1-z$$
$$0 \le z \le 1-\sqrt{x} \qquad 0 \le x \le (1-z)^2$$
$$0 \le x \le 1 \qquad 0 \le z \le 1$$