Lecture 26: Conservation Laws of Electromagnetics

Given charge density ρ and electric field $\vec{E}$ and magnetic field $\vec{B}$ we defined the Poynting Vector

$$\boxed{\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}}$$

energy flux density

$$\boxed{\frac{dW}{dt} = \frac{-d}{dt}\int_{\mathcal{V}} u \, d\tau - \oint_{\partial\mathcal{V}} \vec{S} \cdot d\vec{a}}$$

Poynting's Thm

and the Electromagnetic Stress/Energy tensor

$$\boxed{T_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E_2 \right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij} B^2 \right)}$$

last lecture we derived the force per unit volume as

$$\vec{f} = \sum_{i,j=1}^{3} (\partial_i T_{ij}) \hat{x}_j - \epsilon_0 \frac{\partial}{\partial t}(\vec{E} \times \vec{B})$$

$$\vec{f} = \sum_{i,j=1}^{3} (\partial_i T_{ij}) \hat{x}_j - \epsilon_0 \mu_0 \frac{\partial \vec{S}}{\partial t}$$

thus the force on charge within $\mathcal{V}$ is given by

$$\boxed{\vec{F} = \int_{\partial\mathcal{V}} T(d\vec{a}, \ ) - \mu_0 \epsilon_0 \frac{d}{dt}\int_{\mathcal{V}} \vec{S} \, d\tau}$$

Given static fields the $\int \vec{S} \, d\tau$ has no t-dep. and

$$\boxed{\vec{F} = \int_{\partial\mathcal{V}} T(d\vec{a}, \ )}$$

- T_{11}, T_{22}, T_{33} represent pressures (force per area)
- T_{12}, T_{13}, T_{23} give shears

Conservation of Momentum

$\vec{F} = \dfrac{d\vec{P}_{mech}}{dt}$ is Newton's 2^{nd} Law in terms of the total momentum and total mechanical force on some system.

By analogy, we found total electromagnetic force so,

$$\boxed{\frac{d\vec{P}_{mech}}{dt} = \underbrace{-\epsilon_0\mu_0 \frac{d}{dt}\int_{\mathcal{V}} \vec{S}\, d\tau}_{\text{from momentum in the } \vec{E} \,\&\, \vec{B} \text{ field}} + \underbrace{\int_{\partial\mathcal{V}} T(d\vec{a},\ )}_{\text{momentum per time flowing through } \partial\mathcal{V}}}$$

conservation of momentum governed by this.

$$\boxed{\vec{P} = \mu_0\epsilon_0 \int_{\mathcal{V}} \vec{S}\, d\tau}$$ ← momentum in the $\vec{E}/\vec{B}$ fields within $\mathcal{V}$

$$\boxed{\vec{g} = \mu_0\epsilon_0 \vec{S} = \epsilon_0 (\vec{E}\times\vec{B})}$$ ← momentum density within Electromagnetic field.

<u>In empty space</u> the mechanical momentum is not changing thus

$$0 = -\int_{\mathcal{V}} \frac{\partial \vec{g}}{\partial t} d\tau + \int_{\partial\mathcal{V}} T(d\vec{a},\ )$$

$$= \int_{\mathcal{V}} \left(-\frac{\partial \vec{g}}{\partial t} + \nabla\cdot\overleftrightarrow{T} \right) d\tau$$

$$\nabla\cdot\overleftrightarrow{T} = \sum_{i,j} (\partial_i T_{ij})\, \hat{x}_j$$

$$\therefore \quad \underbrace{\boxed{\nabla\cdot\overleftrightarrow{T} = \frac{\partial\vec{g}}{\partial t}}}_{\text{local conservation of field momentum in the absence of charge}}$$

Dual Role of Poynting Vector

① $\vec{S}$ as energy per unit area time transported by the electromagnetic fields

② $\mu_0 \epsilon_0 \vec{S}$ as momentum per unit volume which is stored in the electromagnetic field

Dual Role of T_{ij}

① electromagnetic stress (force per unit area)

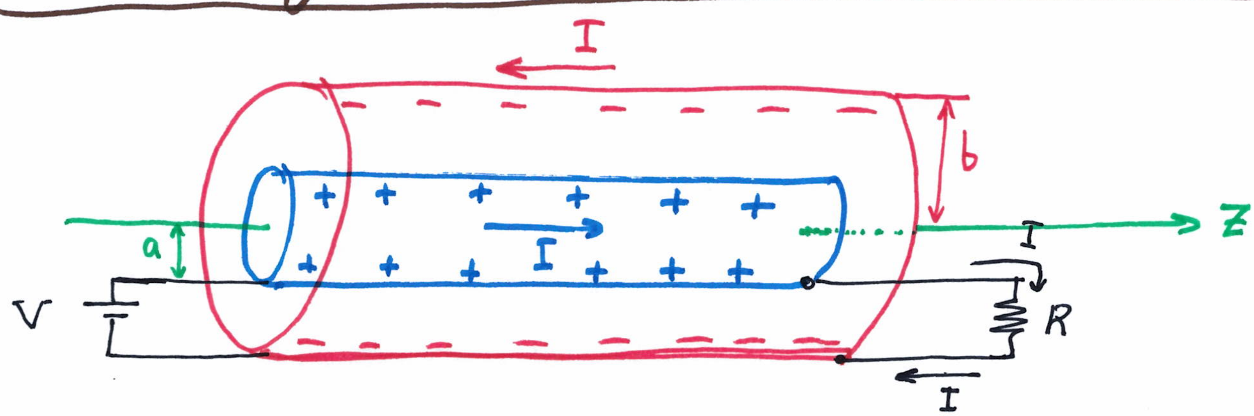
② $-T_{ij}$ describes flow of momentum carried by the fields.

Fuzzy Remark: we soon learn $c = \dfrac{1}{\sqrt{\mu_0 \epsilon_0}}$ and perhaps you recall for light we have the energy and momentum related by $E = PC$. I see this foreshadowed in our observations above for $\vec{S}$

$$\frac{dW}{dt} = \int_{\partial V} \vec{S} \cdot d\vec{a} \qquad \frac{d\vec{P}}{dt} = -\epsilon_0 \mu_0 \frac{d}{dt}\int_V \vec{S}\, d\tau$$

$$= \frac{-1}{c^2}\frac{d}{dt}\int_V \vec{S}\, d\tau$$

E1 A long resistanceless coaxial cable of length l consists of inner conductor (radius a) and outside conductor (radius b). It is connected to a battery at one end and a resistor R at the other. The inner conductor carries uniform charge per unit length λ and a steady current I to the right (the outer conductor has $-\lambda$ and current I to the left). Find electromagnetic momentum stored in fields.



$$\vec{E} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{s}\hat{s}, \qquad \vec{B} = \frac{\mu_0}{2\pi}\frac{I}{s}\hat{\phi}$$

$$\underbrace{\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \left(\frac{\lambda I}{4\pi^2\epsilon_0 s^2}\right)\hat{z}}_{\text{energy flows down cable from voltage source to resistor.}}$$

Power flowing across cross-section of cable

$$P = \int \vec{S}\cdot d\vec{a} = \frac{\lambda I}{4\pi^2\epsilon_0}\int_a^b \frac{2\pi s\, ds}{s^2} = \frac{\lambda I}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right) = IV$$

Since $\vec{E} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{s} = -\hat{s}\frac{\partial V}{\partial s} \rightarrow V(b) - V(a) = V = \frac{\lambda}{2\pi\epsilon_0}[\ln(b) - \ln(a)]$

$$\frac{-\lambda}{2\pi\epsilon_0 s} = \frac{\partial V}{\partial s} \rightarrow V(s) = \frac{-\lambda}{2\pi\epsilon_0}\ln(s) + C$$

$$V = V(a) - V(b) = \frac{-\lambda}{2\pi\epsilon_0}\ln(a) + \frac{\lambda}{2\pi\epsilon_0}\ln(b) = \underline{\frac{\lambda}{2\pi\epsilon_0}\ln(b/a)}.$$

To find momentum in field we integrate $\vec{g} = \mu_0 \epsilon_0 \vec{S}$ over the volume. Since we only have fields within $0 \leq z \leq l$ and $a \leq s \leq b$ and $\vec{g} = \mu_0 \epsilon_0 \left(\frac{\lambda I}{4\pi^2 \epsilon_0 s^2}\right)\hat{z} = \frac{\mu_0 \lambda I}{4\pi^2 s^2}\hat{z}$

$$\vec{p} = \int_{\mathcal{V}} \vec{g}\, d\tau = \left(\int_0^l \int_a^b \int_0^{2\pi} \frac{\mu_0 \lambda I}{4\pi^2 s^2} \cdot s\, d\phi\, ds\, dz\right)\hat{z}$$

$$= \left(\frac{2\pi l \mu_0 \lambda I}{4\pi^2} \int_a^b \frac{ds}{s}\right)\hat{z}$$

$$= \frac{\mu_0 \lambda I l}{2\pi}\left(\ln(b) - \ln(a)\right)\hat{z}$$

$$= \underbrace{\boxed{\epsilon_0 \mu_0 I V l\, \hat{z}}}_{\text{electromagnetic}}$$

momentum of fields in terms of current I and V.

Next, suppose current is decreased by increasing R then the changing $\vec{E}$ induces

$$\vec{E} = \left[\frac{\mu_0}{2\pi}\frac{dI}{dt}\ln(s) + K\right]\hat{z}$$

(derived in LECTURE 22 E9, the "K" is an uninteresting constant)

Hence a force is placed on $\pm\lambda$ of,

$$\vec{F} = \lambda l\left[\frac{\mu_0}{2\pi}\frac{dI}{dt}\ln(a) + K\right]\hat{z} - \lambda l\left[\frac{\mu_0}{2\pi}\frac{dI}{dt}\ln(b) + K\right]\hat{z}$$

$$\boxed{\vec{F} = \frac{-\mu_0 \lambda l}{2\pi}\frac{dI}{dt}\ln(b/a)\,\hat{z}}$$

Then notice the impulse imparted to cable is

$$\vec{P}_{mech} = \int \vec{F}\, dt = \frac{\mu_0 \lambda I}{2\pi}\ln(b/a)\,\hat{z} = \underbrace{\boxed{\epsilon_0 \mu_0 I V l\, \hat{z}}}_{\text{mechanical}}$$

Angular Momentum & Electromagnetic Fields

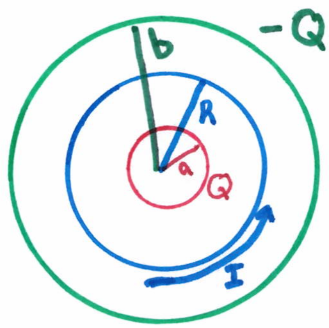
We should consider three densities associated to a given set of electric and magnetic fields

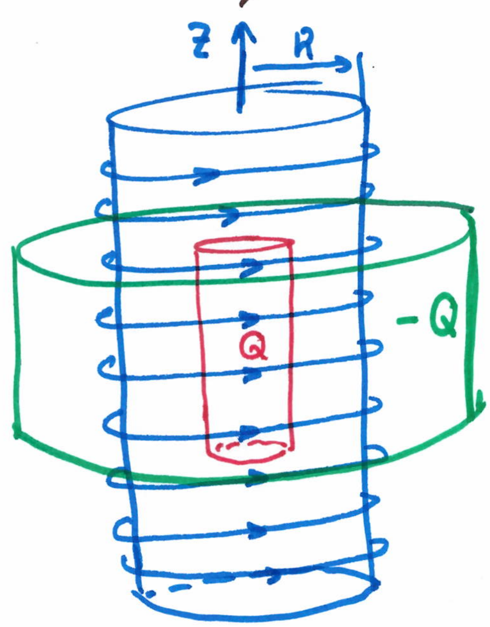
$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) \quad : \text{energy}$$

$$\vec{g} = \epsilon_0 (\vec{E} \times \vec{B}) = \epsilon_0 \mu_0 \vec{S} \quad : \text{momentum}$$

$$\vec{\ell} = \vec{r} \times \vec{g} = \epsilon_0 [\vec{r} \times (\vec{E} \times \vec{B})] \quad : \text{angular momentum}$$

E2 Imagine very long solenoid with radius R, n turns per length and current I. Coaxial with the solenoid are two cylindrical nonconducting shells of length ℓ, one inside at $a < R$ and the other outside at $b > R$. Inside carries Q, outside carries $-Q$





When the current in solenoid at $s = R$ is gradually reduced then the cylinders begin to rotate (we discussed this in previous lecture) (GRIFFITH'S Ex. 7.9)

E2 continued

Let us determine the angular momentum stored in the electromagnetic field initially,

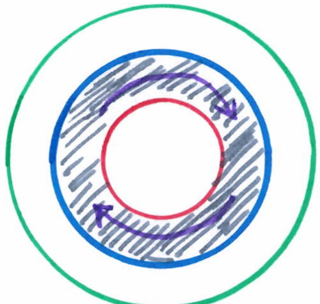
$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0}\frac{\hat{s}}{s} = \left(\frac{Q}{2\pi\epsilon_0 \ell}\frac{1}{s}\right)\hat{s} \quad (\text{for } a < s < b)$$

Here $\lambda = \frac{Q}{\ell}$ from the charge Q uniformly distributed along the cylinder at $s = a$. The magnetic field of a long solenoid is given by

$$\vec{B} = \mu_0 n I \hat{z} \quad (\text{for } s < R)$$

Now we calculate the momentum density $\vec{g} = \mu_0\epsilon_0 \vec{E}\times\vec{B}$

$$\vec{g} = \mu_0\epsilon_0\left(\frac{Q\hat{s}}{2\pi\epsilon_0 \ell s}\right)\times\left(\mu_0 n I\hat{z}\right) = \underbrace{\frac{-\mu_0 n I Q}{2\pi\ell s}\hat{\phi}}_{\text{for } a<s<R}$$



($\vec{g}$ goes CW in shaded region)

Angular momentum $\vec{\ell} = \vec{r}\times\vec{g}$ is in $-z$-direction since $\hat{s}\times\hat{\phi} = \hat{z} \Rightarrow \hat{s}\times(-\hat{\phi}) = -\hat{z}$

$$\vec{\ell} = \frac{-\mu_0 n I Q}{2\pi\ell s}(s\hat{s})\times\hat{\phi} = \underbrace{\frac{-\mu_0 n I Q}{2\pi\ell}\hat{z}}_{\text{constant throughout } 0\le z\le\ell,\ a\le s\le R}$$

Then the total angular momentum is found by $\vec{L} = \int \vec{\ell}\, d\tau$

$$\boxed{\vec{L} = \ell\pi(R^2 - a^2)\left(\frac{-\mu_0 n I Q}{2\pi\ell}\hat{z}\right) = \frac{-1}{2}\mu_0 n I Q (R^2 - a^2)\hat{z}}$$

E2 continued

When the current is turned off then the changing B-field induces a cirumferential electric field via Faraday's Law

$$E = \begin{cases} -\frac{1}{2}\mu_0 n \frac{dI}{dt}\frac{R^2}{s}\hat{\phi} & : s > R \\ -\frac{1}{2}\mu_0 n \frac{dI}{dt} s\,\hat{\phi} & : s < R \end{cases}$$

Consequently, the torque on the outside cylinder,

$$\vec{N}_b = \vec{r} \times (-Q\vec{E}) = \frac{1}{2}\mu_0 n Q R^2 \frac{dI}{dt}\hat{z}$$

Likewise on the inside cylinder,

$$\vec{N}_a = -\frac{1}{2}\mu_0 n Q a^2 \frac{dI}{dt}\hat{z}$$

We calculate change in mechanical angular momentum by integrating torque,

$$\vec{L}_b = \frac{\hat{z}}{2}\mu_0 n Q R^2 \int_I^0 \frac{dI}{dt} dt = -\frac{1}{2}\mu_0 n I Q R^2 \hat{z}$$

$$\vec{L}_a = -\frac{1}{2}\mu_0 n Q a^2 \hat{z}\int_I^0 dI = \frac{1}{2}\mu_0 n I Q a^2 \hat{z}$$

Therefore, angular momentum is conserved!

$$\boxed{\vec{L}_{electromagnetic} = -\frac{1}{2}\mu_0 n I Q (R^2 - a^2)\hat{z} = \vec{L}_a + \vec{L}_b = \vec{L}_{mechanical}}$$

angular momentum is conserved, it changes form from electromagnetic $\longrightarrow$ mechanical as the current is reduced from I to 0.