Lecture 26 : Intuition on Inverse Function Th$^{\underline{m}}$

Given $F: \mathbb{R}^n \to \mathbb{R}^n$ we can ask if there exists a local inverse for F at $x_0 \in \mathbb{R}^n$? Algebraically, can we solve $F(x) = y$ for $x = F^{-1}(y)$.

Example 1: $F(x,y) = (3 + e^x - e^y, 3e^x + e^y) = (a,b)$

Notation: $F: \mathbb{R}_{xy} \to \mathbb{R}_{ab}$ we have:

$$a = 3 + e^x - e^y$$
$$b = 3e^x + e^y$$

Then $a + b = 3 + 4e^x \Rightarrow e^x = \frac{a+b-3}{4}$

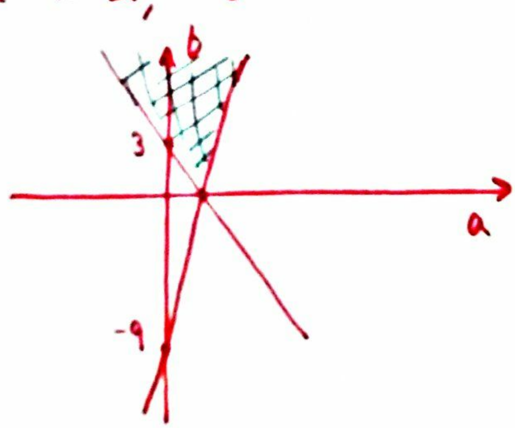
thus $x = \ln\left(\frac{1}{4}(a+b-3)\right)$. Thus

$b = \frac{3}{4}(a+b-3) + e^y \Rightarrow e^y = \frac{1}{4}(b - 3a + 9)$

Hence $y = \ln\left(\frac{1}{4}(b-3a+9)\right)$. Summary,

$$\boxed{F^{-1}(a,b) = \left(\ln(a+b-3) - \ln(4), \ln(b-3a+9) - \ln(4)\right)}$$

Remark: If $F(x,y) = (x^3 + y^2 + y^3, xy)$ then it would not be so easy to calculate $F^{-1}(a,b)$. Yet, we'd like to know at least in principle if it's possible... Incidentally notice $a+b-3 > 0$ and $b - 3a + 9 > 0$ are necessary condition to calculate $F^{-1}(a,b)$. We can visualize these, $b > 3-a$ & $b > 3a - 9$



$$3 - a = 3a - 9$$
$$4a = 12$$
$$\underline{a = 3}$$

The problem of solving $F(x) = y$ for x near x_o is naturally approximated by solving $L_F^{x_o}(x) = y$ for x near x_o. Consider,

$$F(x_o) + J_F(x_o)(x - x_o) \approx y$$

$$\Rightarrow J_F(x_o)(x - x_o) = y - F(x_o)$$

Apparently we need $J_F(x_o)^{-1}$ to exist ($n \times n$ matrix inverse)
Then we find,

$$\boxed{x = x_o + (J_F(x_o))^{-1}(y - F(x_o))}$$

It seems $\det(J_F(x_o)) \neq 0$ will allow us to find local inverse near enough to x_o. In fact pgs. 181-185 of Edward's ADVANCED CALCULUS OF SEVERAL VARIABLES, show the contraction mapping Th^{m} extends to $\mathbb{R}^n$ and the argument given for $n=1$ local inverse via Newton's Method motivated sequence likewise extends (there are technical details that require some time to fully absorb...) the result is:

Th^{m} (3.3.) Suppose $F: \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable in a nbhd W of the point a, with matrix $J_F(a)$ nonsingular, Then F is locally invertible – $\exists$ nbhds $U \subseteq W$ of a and V of $b = F(a)$ and a 1-1 continuously diff. mapping $g: V \to W$ such that

$$g(F(x)) = x \quad \forall x \in U$$
$$F(g(y)) = y \quad \forall y \in V.$$

In particular, g is the limit of sequence $\{g_n\}_0^\infty$ of successive approximations defined by

$$g_o(y) = a, \quad g_{n+1}(y) = g_n(y) - J_F(a)^{-1}[F(g_n(y)) - y]$$

for $y \in V$

<u>Example 1 continued</u>

$$F(x,y) = (3+e^x - e^y,\ 3e^x + e^y)$$

$$J_F = \begin{bmatrix} e^x & -e^y \\ 3e^x & e^y \end{bmatrix}$$

$$\det(J_F) = e^x e^y + 3e^x e^y = 4e^{x+y} \neq 0 \quad \forall (x,y) \in \mathbb{R}^2$$

Hence F is locally invertible at any point.

<u>Example 2</u>

$$F(x,y,z) = (1+x^2,\ 1+y^2,\ 3z) = (a,b,c)$$

$$J_F = \begin{bmatrix} 2x & 0 & 0 \\ 0 & 2y & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\det(J_F) = 12xy$$

For $xy \neq 0$ we have the existence of local inverse.

Notice $a = 1+x^2$, $b = 1+y^2$, $c = 3z$

$$x = \pm\sqrt{a-1} \quad y = \pm\sqrt{b-1}, \quad z = \tfrac{1}{3}c$$

when $x = 0$ or $y = 0$ we get $\pm$ sol^{n}s at once

However, away from $x = 0$ and $y = 0$ we can select $+$ or $-$ and so construct $F^{-1}(a,b,c)$

<u>Example 3:</u>

$$F(x,y) = (x^2 - y^2,\ 2xy) \hookrightarrow J_F = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$$

$$\det J_F = 4(x^2+y^2) \Rightarrow$$ local inverse exists at points with x or y nonzero.