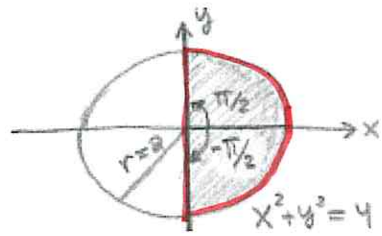
Lecture 27b: CHANGE OF VARIABLES

Example 6.5.12. *Let* $R = \{(x,y) | x^2+y^2 \leq 4, x \geq 0\}$. *Convert this region to Polars and integrate* $f(x,y) = \sqrt{4-x^2-y^2}$.



$0 \leq r \leq 2$

$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

It is clear from the plot above that $S = \{(r,\theta) \mid 0 \leq r \leq 2, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\}$. *Note, we allow* θ *to range outside* $[0, 2\pi)$ *for our convenience. If we insisted on using only* $[0,2\pi)$ *then the* θ*-domain would be built from disconnected intervals* $[0,\pi/2]$ *and* $[3\pi/2, 2\pi)$. *Thankfully, for problems of integration we consider, we are free to use the more convenient domain. I should warn the reader this difficulty cannot be avoided in complex analysis and ultimately leads to some rather interesting results. I digress, let's get back to the integration:*

$$\begin{aligned}
\iint_R \sqrt{4-x^2-y^2}\,dA &= \int_0^2 \int_{-\pi/2}^{\pi/2} \sqrt{4-r^2\cos^2\theta - r^2\sin^2\theta}\,\underline{r\,d\theta\,dr} \\
&= \int_0^2 \int_{-\pi/2}^{\pi/2} r\sqrt{4-r^2}\,d\theta\,dr \\
&= \int_0^2 \pi r\sqrt{4-r^2}dr \\
&= -\frac{\pi}{2}\frac{2}{3}(4-r^2)^{3/2}\Big|_0^2 \\
&= -\frac{\pi}{3}[0-(2^2)^{3/2}] \\
&= \boxed{\frac{8\pi}{3}.}
\end{aligned}$$

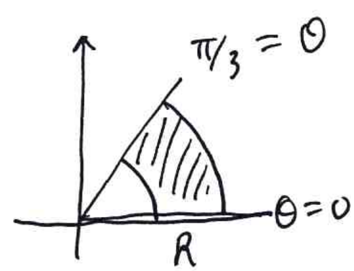
$\boxed{dA = r\,d\theta\,dr}$

//

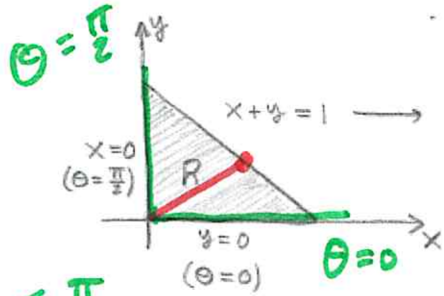
$$\int_0^1 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{4-x^2-y^2}\,dy\,dx$$

Example 6.5.13. *Let* $f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$ *find* $\iint_R f(x,y)dA$ *where* R *is the region in xy-plane with* $1 \leq r \leq 2$ *and* $0 \leq \theta \leq \pi/3$.

$$\begin{aligned}
\iint_R \frac{1}{\sqrt{x^2+y^2}}dA &= \int_0^{\pi/3} \int_1^2 \frac{1}{r} r\,dr\,d\theta \\
&= \left(\int_0^{\pi/3} d\theta\right)\left(\int_1^2 dr\right) \\
&= \boxed{\frac{\pi}{3}.}
\end{aligned}$$



Example 6.5.14. The wrong way to calculate the area of a triangle: *Find the area of the triangle bounded by $x = 0, y = 0$ and $x + y = 1$ using polar coordinates. It's fairly easy to see that $0 \leq \theta \leq \pi/2$ on R, however bounding r requires thought,*



$0 \leq \theta \leq \frac{\pi}{2}$

$$x+y=1 \longrightarrow r\cos\theta + r\sin\theta = 1$$

$$r(\cos\theta + \sin\theta) = 1$$

$$r = \frac{1}{\cos\theta + \sin\theta}$$

$$\Rightarrow 0 \leq r \leq \frac{1}{\cos\theta + \sin\theta}$$

$r_{in} \leq r \leq r_{out}$

functions of θ

This is a less trivial polor region, we must put the integration over dr first since its bounds are θ-dependent.

$$\begin{aligned}
Area(R) &= \int_0^{\pi/2} \int_1^{\frac{1}{\cos\theta+\sin\theta}} r\, dr\, d\theta \\
&= \int_0^{\pi/2} \frac{1}{2}\left(r^2 \Big|_0^{1/[\cos\theta+\sin\theta]} \right) d\theta \\
&= \frac{1}{2}\int_0^{\pi/2} \frac{d\theta}{(\cos\theta+\sin\theta)^2} \\
&= \frac{1}{2}\int_0^{\pi/2} \frac{1}{2}\csc^2(\theta+\pi/4)\Big|_0^{\pi/2} \\
&= \frac{-1}{4}\cot(\theta+\pi/4)\Big|_0^{\pi/2} \\
&= \frac{-1}{4}\Big(\cot(3\pi/4) - \cot(\pi/4)\Big) \\
&= \frac{-1}{4}(-1-1) \\
&= \boxed{\frac{1}{2}}
\end{aligned}$$

$\cos\theta + \sin\theta = \sin(\theta + \pi/4)$

At $\star$ we noticed $\sin(\theta+\pi/4) = \sin\theta\cos\pi/4 + \sin\pi/4\cos\theta = \frac{1}{\sqrt{2}}(\sin\theta+\cos\theta)$ *thus* $\sin\theta + \cos\theta = \sqrt{2}(\sin(\theta+\pi/4)$. *This is a horrible method to find the area of a triangle. But, it illustrates a general principle which is that coordinates should be chosen to fit the problem. Obviously this problem is far more natural in Cartesians coordinates.*

Example 6.5.15. *Let $R = \{(x,y) | 9x^2 + 4y^2 \leq 36\}$. Calculate $\iint_R x^2 dA$.*

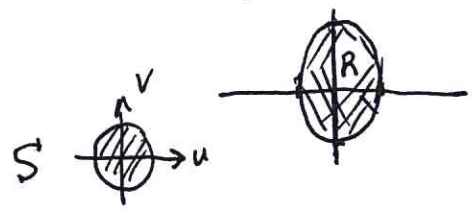
Solution: *Let $x = 2u$ and $y = 3v$ and observe that $36 = 4(3v) = 9x^2 + 4y^2 = 9(2u)^2 + 4(3v)^2 = 36u^2 + 36v^2$ hence the equation of the ellipse is transformed to the unit-circle $u^2 + v^2 = 1$ in uv-space. Note,*

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$$

$$\frac{x^2}{4} + \frac{y^2}{9} \leq 1$$

Let $S = \{(u,v) \mid u^2 + v^2 \leq 1\}$ and calculate:

$$\iint_R x^2 dA = \iint_S (2u)^2 \frac{\partial(x,y)}{\partial(u,v)} du\, dv$$
$$= \iint_S 24u^2 du\, dv$$



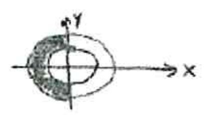
It is useful to change coordinates once more: let $u = r\cos\theta$ and $v = r\sin\theta$ then S transforms to $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 1$ in $r\theta$-space.

$$[0,1] \times [0, 2\pi]$$

$$\begin{aligned}
\iint_R x^2 dA &= \int_0^1 \int_0^{2\pi} 24r^2 \cos^2\theta r d\theta\, dr \\
&= \int_0^1 24r^3 dr \int_0^{2\pi} \frac{1}{2}(1 + \cos(2\theta))d\theta \\
&= \left(6r^4\Big|_0^1\right)\left(\frac{1}{2}(1 + \sin(2\theta))\Big|_0^{2\pi}\right) \\
&= (6-0)\left[\frac{1}{2}\left(2\pi + \frac{\sin(4\pi)}{2}\right) - \frac{1}{2}\left(0 + \frac{1}{2}\sin(0)\right)\right] \\
&= \boxed{6\pi.}
\end{aligned}$$

Example 6.5.16. *Let R be the region in the xy-plane with $x \leq 0$ and $1 \leq x^2 + y^2 \leq 4$. Calculate $\iint_R (x+y)dA$ by changing the integral to polar coordinates.*

Solution: *to begin R is easily seen to have $1 \leq r \leq 2$ and $\frac{\pi}{2} \leq \theta\frac{3\pi}{2}$. See the crude sketch below:*



$1 \leq r \leq 2$
$\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$

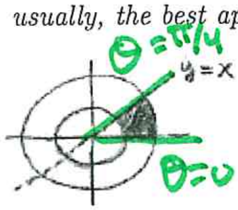
$$\begin{aligned}
\iint_R (x+y)dA &= \int_{\pi/2}^{3\pi/2} \int_1^2 (r\cos\theta + r\sin\theta) r dr\, d\theta \\
&= \int_{\pi/2}^{3\pi/2} (\cos\theta + \sin\theta) d\theta \int_1^2 r^2 dr \\
&= \left([\sin\theta - \cos\theta]\Big|_{\pi/2}^{3\pi/2}\right) \frac{1}{3}r^3\Big|_1^2 \\
&= (-1-1)\left(\frac{1}{3}(8-1)\right) = \boxed{-\frac{14}{3}.}
\end{aligned}$$

Example 6.5.17. *Let* $R = \{(x,y) \mid 1 \leq x^2+y^2 \leq 4, 0 \leq y \leq x\}$. *Calculate* $\iint_R \tan^{-1}(y/x)dA$ *by changing the integration to polar coordinates.*

$1 \leq r^2 \leq 4$

$\underline{1 \leq r \leq 2}.$

Solution: *usually, the best approach is to draw a picture. Consider,*



$R = \{(r,\theta) \mid 1 \leq r \leq 2, \ 0 \leq \theta \leq \frac{\pi}{4}\}$

$\tan^{-1}(y/x) = \tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta}\right) = \tan^{-1}(\tan\theta) = \theta$

Therefore,

$$\iint_R \tan^{-1}(y/x)dA = \int_0^{\pi/4}\int_1^2 \theta r\, dr\, d\theta$$

$dA = r\,dr\,d\theta$

$$= \frac{1}{2}\theta^2\Big|_0^{\pi/4}\frac{1}{2}r^2\Big|_1^2$$

$$= \frac{1}{4}\left(\frac{\pi^2}{16}\right)(4-1)$$

$$= \boxed{\frac{3\pi^2}{64}.}$$

$z = 18 - 2(x^2+y^2) = 18 - 2r^2$

2

Example 6.5.18. *Find volume bounded by* $z = 18 - 2x^2 - xy^2$ *and* $z = 0$.

Solution: *The double integral will yield the volume. First convert to polars,*

$$z = 18 - 2r^2 \quad\Rightarrow\quad 9 = r^2 \quad\Rightarrow\quad r = 3.$$

the surface intersection with $z = 0$ *in a circle* $r = 3$ *and* $R = \{(r,\theta) \mid 0 \leq r \leq 3, 0 \leq \theta \leq 2\pi\}$. *Therefore, the volume is found by calculating*

$$V = \iint_R z dA = \int_0^{2\pi}\int_0^3 (18-2r^2)r dr\, d\theta$$

$$= (2\pi)\left(9r^2 - \frac{2}{4}r^4\right)\Big|_0^3$$

$$= (2\pi)(81 - \frac{1}{2}(81))$$

$$= \boxed{81\pi.}$$

Example 6.5.19. *Find volume bounded by $z = \sqrt{x^2+y^2}$ and $x^2+y^2+z^2 = 1$. This is a cone bounded by a sphere. The equations are symmetric in x and y hence we deduce $0 \leq \theta \leq 2\pi$. The intersection of these surfaces has $x^2+y^2+(\sqrt{x^2+y^2})^2 = 1$ hence $2r^2 = 1$ from which we find $r = 1/\sqrt{2}$ on the intersection. It follows that $0 \leq r \leq 1/\sqrt{2}$. Naturally, you could use a three-dimensional graphing utility for further confirmation of our claim. To set up the volume, we need to identify that the shape has $\sqrt{x^2+y^2} \leq z \leq \sqrt{1-x^2-y^2}$ for*

$$\begin{aligned} dV &= (z_{top} - z_{base})\, dA \leftarrow \textit{(typital infinitesimal volume.)} \\ &= \left(\sqrt{1-x^2-y^2} - \sqrt{x^2+y^2}\right) dx\, dy \\ &= \left(\sqrt{1-r^2} - r\right) r\, dr\, d\theta \end{aligned}$$

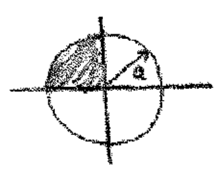
Thus,

$$\begin{aligned} V = \int dV &= \int_0^{2\pi}\int_0^{1/\sqrt{2}} (r\sqrt{1-r^2} - r^2)\,dr\, d\theta \\ &= 2\pi\left(\frac{-1}{3}(1-r^2)^{3/2} - \frac{1}{3}r^3\bigg|_0^{\frac{\sqrt{2}}{2}}\right) \\ &= \frac{-2\pi}{3}\left(\left(\frac{1}{2}\right)^{3/2} + \left(\frac{1}{\sqrt{2}}\right)^3 - 1\right) \\ &= \boxed{\frac{\pi}{3}(2-\sqrt{2})} \end{aligned}$$

$$\iiint dV = \int_0^{2\pi}\int_0^{1/\sqrt{2}}\int_r^{\sqrt{1-r^2}} r\, dz\, dr\, d\theta$$

Example 6.5.20. *Convert $\int_0^a \int_{-\sqrt{a^2-y^2}}^0 x^2 y\, dx\, dy$ to polar coordinates and calculate the integral.*

Solution: *from the given integral we deduce the integration region is given by $0 \leq y \leq a$ and $-\sqrt{a^2-y^2} \leq x \leq 0$. It follows the region is the top left quarter of the disk $x^2+y^2 \leq a^2$:*



$0 \leq r \leq a$

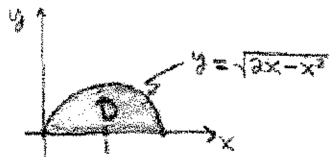
$\frac{\pi}{2} \leq \theta \leq \pi$

Therefore,

$$\begin{aligned} \int_{\pi/2}^{\pi}\int_0^a (r^2\cos^2\theta)(r\sin\theta) r\, dr\, d\theta &= \int_{\pi/2}^{\pi} \cos^2\theta \sin\theta d\theta \int_0^a r^4 dr \\ &= \left(\frac{-1}{3}\cos^3\theta\bigg|_{\pi/2}^{\pi}\right)\left(\frac{a^5}{5}\right) \\ &= \frac{-1}{3}(-1)(a^5/5) = \boxed{a^5/15.} \end{aligned}$$

Example 6.5.21. *Convert* $\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx$ *to polar coordinates and calculate the integral.*

Solution: *let* D *be the region of integration. If* $(x,y) \in D$ *then we may read from the given integral that* $0 \leq y \leq \sqrt{2x-x^2} = \sqrt{x(2-x)}$ *and* $0 \leq x \leq 2$ *for the integration region. This is pictured below:*



$\Rightarrow y^2 = 2x - x^2$
$\Rightarrow x^2 - 2x + y^2 = 0$
$\Rightarrow (x-1)^2 + y^2 = 1$ (it's a circle of radius one at $(1,0)$)

it should be clear that D *has* $0 \leq \theta \leq \pi/2$. *It is also clear that the bound on* r *must depend on* θ *since we have differing radii for differing* θ *(for example,* $r = 2$ *gives* $\theta = 0$ *while* $r = 0$ *gives* $\theta = \pi/2$*). We need to convert* $y = \sqrt{2x-x^2}$ *to a more useful form. Note that* $x^2 - 2x + y^2 = 0$ *yields* $x^2 + y^2 = 2x$. *However, this yields* $r^2 = 2r\cos\theta$ *hence we obtain the equation of the half-circle as* $r = 2\cos\theta$ *for* $0 \leq \theta \leq \pi/2$. *Therefore,*

$$\begin{aligned}
\iint_D \sqrt{x^2+y^2}\,dA &= \int_0^{\pi/2} \int_0^{2\cos\theta} r^2\,dr\,d\theta \\
&= \int_0^{\pi/2} \frac{8}{3}\cos^3\theta\,d\theta \\
&= \int_0^{\pi/2} \frac{8}{3}(1-\sin^2\theta)\cos\theta\,d\theta \\
&= \frac{8}{3}\left(\sin\theta - \frac{1}{3}\sin^3\theta\right)\Big|_0^{\pi/2} \\
&= \frac{8}{3}\left(1 - \frac{1}{3}\right) = \boxed{\frac{16}{9}}
\end{aligned}$$

Example 6.5.22. *Let D_a be disk of radius a centered at origin, define*

$$
\begin{aligned}
I = \iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dA &= \lim_{a\to\infty}\int_{D_a} e^{-x^2-y^2}\,dA \\
&= \lim_{a\to\infty}\int_0^{2\pi}\int_0^a e^{-r^2}\,r\,dr\,d\theta \\
&= \lim_{a\to\infty}\left((2\pi)\frac{-1}{2}e^{-r^2}\Big|_0^a\right) \\
&= \lim_{a\to\infty}\pi(-e^{-a^2}+1) \\
&= \pi
\end{aligned}
$$

Alternatively, we could calculate the integral of $e^{-x^2-y^2}$ by a limit of rectangular integrals: let $S_a = [-a,a]\times[-a,a]$

$$
\begin{aligned}
I = \int_{\mathbb{R}^2} e^{-x^2-y^2}\,dA &= \lim_{a\to\infty}\iint_{S_a} e^{-x^2-y^2}\,dA \\
&= \lim_{a\to\infty}\int_{-a}^{a}\int_{-a}^{a} e^{-x^2}e^{-y^2}\,dx\,dy \\
&= \lim_{a\to\infty}\left(\int_{-a}^{a} e^{-x^2}\,dx\right)\left(\int_{-a}^{a} e^{-y^2}\,dy\right) \\
&= \lim_{a\to\infty}\left(\int_{-a}^{a} e^{-x^2}\,dx\right)\lim_{a\to\infty}\left(\int_{-a}^{a} e^{-y^2}\,dy\right) \\
&= \left[\lim_{a\to\infty}\left(\int_{-a}^{a} e^{-x^2}\,dx\right)\right]^2 = \pi
\end{aligned}
$$

Therefore, we discover an interesting result. Assuming both limiting schemes agree,

$$
\lim_{a\to\infty}\left(\int_{-a}^{a} e^{-x^2}\,dx\right) = \sqrt{\pi}. \longrightarrow \boxed{\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}}
$$

Example 6.5.23. *A slight modification of the previous examples shows $\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}}$. Assume that $\frac{\partial}{\partial a}\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \int_{-\infty}^{\infty}\frac{\partial}{\partial a}e^{-ax^2}\,dx$ to derive many nice formulas for integrals of the form:*

$$
\int_{-\infty}^{\infty} z^n e^{-az^2}\,dz.
$$

If you want to know a lot more about the idea of this example, you might search for the excellent paper by Keith Conrad DIFFERENTIATING UNDER THE INTEGRAL SIGN. In fact, you'll find many interesting expository papers at Professor Conrad's website. "Blurbs"

6.6 triple integrals involving coordinate change

The change of variables theorem for triple integrals and higher integrals is essentially the same as we just saw for double integrals. We begin by extending the definition of the Jacobian to three variables:

Definition 6.6.1.

Let $T(y, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w))$ be a differentiable function from $\mathbb{R}^3 \to \mathbb{R}^3$, the Jacobian of T is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Example 6.6.2. *Cylindrical coordinates:* $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \det\begin{bmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{bmatrix} = \det\begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = (r\cos^2\theta + r\sin^2\theta) = \boxed{r}$$

$$\boxed{dV = \underline{dx\,dy\,dz} = \underline{r\,dr\,d\theta}\,dz}$$

Example 6.6.3. *Spherical coordinates:* $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$, $z = \rho\cos\phi$. *where* $0 \le \theta \le 2\pi$ *and* $0 \le \phi \le \pi$ *and* $\rho^2 = x^2 + y^2 + z^2$*:*

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \det\begin{bmatrix} x_\rho & x_\theta & x_\phi \\ y_\rho & y_\theta & y_\phi \\ z_\rho & z_\theta & z_\phi \end{bmatrix}$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \rho^2\sin\phi$$

$$\boxed{dV = \rho^2\sin\phi\, d\rho\, d\phi\, d\theta}$$

$$= \det\begin{bmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{bmatrix}$$

$$\begin{aligned}
&= \cos\theta\sin\phi(\rho\cos\theta\sin\phi)(-\rho\sin\phi) - \sin\theta\sin\phi(\rho^2\sin\theta\sin^2\phi) \\
&\quad + \cos\phi(-\rho^2\sin^2\theta\sin\phi\cos\phi - \rho^2\cos^2\theta\sin\phi\cos\phi) \\
&= -\rho^2\sin^3\phi(\cos^2\theta + \sin^2\theta) - \rho^2\cos^2\phi(\sin^2\theta + \cos^2\theta)\sin\phi \\
&= -\rho^2\sin\phi(\sin^2\phi + \cos^2\phi) \\
&= \boxed{-\rho^2\sin\phi}
\end{aligned}$$

Notice that $\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \rho^2\sin\phi$ *thanks to the fact that swapping a pair of columns in a determinant changes the sign of the result.*

Example 6.6.4. *Let $x = e^{u-v}, y = e^{u+v}, z = e^{u+v+w}$ find Jacobian. By definition,*

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(u,v,w)} &= \det \begin{bmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{bmatrix} \\
&= \det \begin{bmatrix} e^{u-v} & -e^{u-v} & 0 \\ e^{u+v} & e^{u+v} & 0 \\ e^{u+v+w} & e^{u+v+w} & e^{u+v+w} \end{bmatrix} \\
&= e^{u+v+w}(e^{u-v}e^{u+v} + e^{u+v}e^{u-v}) \\
&= e^{u+v+w}(e^{2u}) \\
&= e^{3u+v+w}
\end{aligned}$$

I should mention, Proposition 6.5.6 generalizes to three or more variables.

Example 6.6.5. *Let $x = \frac{u}{v}, y = \frac{v}{w}, z = \frac{w}{u}$ then*

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(u,v,w)} &= det \begin{bmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{bmatrix} \\
&= det \begin{bmatrix} 1/v & -u/v^2 & 0 \\ 0 & 1/w & -v/w^2 \\ -w/u^2 & 0 & 1/u \end{bmatrix} \\
&= \frac{1}{uvw} - \frac{uvw}{u^2v^2w^2} \\
&= \boxed{0.}
\end{aligned}$$

The transformation studied in the example above would not be allowed if we wish to use the change of variables integration Theorem. The everywhere vanishing Jacobian suggests the transformation is not invertible. The theorem below is completely analogous to what we saw already in $n = 2$:

Theorem 6.6.6. *Coordinate Change in Triple Integrals:*

Suppose $T : R \subset \mathbb{R}^3 \to S \subset \mathbb{R}^3$ is a differentiable, mostly invertible mapping where $T(y, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w))$. Also, suppose f is continuous on S, and $T(S) = R$ then

$$\iiint_S f(x, y, z)\, dx\, dy\, dz = \iiint_R f(T(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw$$

Notice the notation $f(T(u, v, w))$ is simply notation for saying that f is to be written in terms of u, v, w as indicated by the formulas $x = x(u, v, w)$, $y = y(u, v, w)$ and $z(u, v, w)$. The following pair of examples give the most common applications of the change of variables theorem for triple integrals in this course:

Example 6.6.7. Cylindrical change of variables: *in view of Example 6.6.2 and the change of variables theorem:*

$$\iiint_R f(x, y, z) dV = \iiint_S f(r\cos\theta, r\sin\theta, z) \cdot r dr\, d\theta\, dz$$

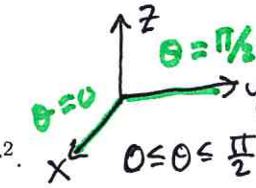
The set S is simply R expressed in cylindrical coordinates.

Example 6.6.8. Spherical change of variables: *in view of Example 6.6.3 and the change of variables theorem:*

$$\iiint_R f(x,y,z)dV = \iiint_S f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)\cdot\rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

The set S is simply R expressed in spherical coordinates.

Example 6.6.9. *Evaluate $\iiint_E (x^3+xy^2)dV$ where E is the solid in the first octant which lies beneath the paraboloid $z = 1-x^2-y^2$.*



Solution: *This problem suggests a cylindrical approach, notice*

$$x^3+xy^2 = x(x^2+y^2) = xr^2 = r^3\cos\theta \qquad \textit{whereas} \qquad z = 1-x^2-y^2 = 1-r^2.$$

Also, the "first octant" is defined by $x\geq 0, y\geq 0$ and $z \geq 0$. On $z=0$ we find the intersection of $z = 1-r^2 = 0 \Rightarrow r^2 = 1$. Recall $dV = r\,dr\,d\theta\,dz$ for cylindrical coordinates. Also, we arrange the iterated bounds to reflect the description we found for E in cylindrical coordinates: $0\leq z\leq 1-r^2$ for $0\leq r\leq 1$ where $0\leq\theta\leq\pi/2$. Therefore,

$$\begin{aligned}\iiint_E (x^3+xy^2)dV &= \int_0^{\pi/2}\int_0^1\int_0^{1-r^2}(r^2\cos\theta)r\,dz\,dr\,d\theta\\ &= \int_0^{\pi/2}\cos\theta d\theta\int_0^1 r^3(1-r^2)dr\\ &= \sin\theta\Big|_0^{\pi/2}\left(\frac{r^4}{4}-\frac{r^6}{6}\right)\Big|_0^1\\ &= (1-0)\left(\frac{1}{4}-\frac{1}{6}\right) = \frac{2}{24} = \boxed{\frac{1}{12}}.\end{aligned}$$

Example 6.6.10. *Consider $f(x,y,z) = \frac{\exp(\sqrt{x^2+y^2+z^2})}{x^2+y^2+z^2}$. Find $\iiint_E f dV$ where $E = \{(x,y,z)|1\leq x^2+y^2+z^2\leq 9\}$.*

e^ρ/ρ^2

$1 \leq \rho^2 \leq 9$

Solution: *clearly this problem is best approached in spherical coordinates. Begin by noting that E may be described in spherical coordinates by $1\leq\rho\leq 3$ and $0\leq\theta\leq 2\pi$ with $0\leq\phi\leq\pi$. The formula for the integrand is likewise transformed to e^ρ/ρ^2. Finally, the Jacobian for spherical coordinates tells us how the volume element transforms to $dV = \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$*

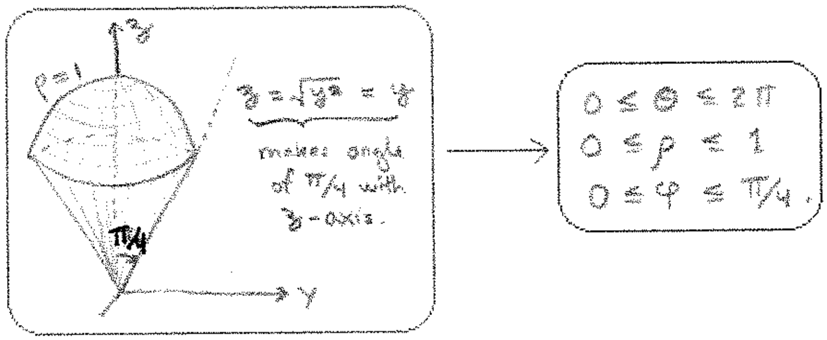
$$\begin{aligned}\iiint_E \frac{\exp(\sqrt{x^2+y^2+z^2})}{x^2+y^2+z^2}dx\,dy\,dz &= \int_0^\pi\int_0^{2\pi}\int_1^3\frac{1}{\rho^2}e^\rho\rho^2\sin\phi\,d\rho\,d\theta\,d\phi\\ &= \left(\int_0^\pi\sin\phi d\phi\right)\left(\int_0^{2\pi}d\theta\right)\left(\int_1^3 e^\rho d\rho\right)\\ &= (-\cos\pi+\cos 0)(2\pi)(e^3-e^1)\\ &= \boxed{4\pi(e^3-e)}\end{aligned}$$

Example 6.6.11. *Find the volume and center of mass of the region E which is above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = 1$. We assume that E has a constant density*[10] δ.

Solution: *the center of mass is found by taking a total-mass-weighted average of x, y, z over the given solid. In particular, essentially by definition, the center of mass at (x_{cm}, y_{cm}, z_{cm}) is given by:*

$$x_{cm} = \frac{1}{M}\iiint_E x\delta\, dV \qquad y_{cm} = \frac{1}{M}\iiint_E y\delta\, dV, \qquad z_{cm} = \frac{1}{M}\iiint_E z\delta\, dV$$

where $\delta = \frac{dM}{dV}$ gives $M = \int dM = \iiint_E \delta dV$. In all four of the integrals we face, a spherical coordinate choice is convenient. Let us begin by picturing the solid:



Thus, pulling out the constant δ,

$$\begin{aligned} M &= \delta \int_0^{\pi/4}\int_0^1\int_0^{2\pi} \rho^2 \sin\phi \, d\theta\, d\rho\, d\phi \\ &= \delta \int_0^1 \rho^2 d\rho \int_0^{\pi/4} \sin\phi\, d\phi \int_0^{2\pi} d\theta \\ &= \delta\left(\frac{1}{3}\right)\Big(-\cos(-\pi/4)+1\Big)2\pi \\ &= \boxed{\frac{2\pi\delta}{3}\left(1-\frac{1}{\sqrt{2}}\right)} \end{aligned}$$

Since δ is a constant density it follows that $\delta = \frac{dM}{dV} = \frac{M}{V}$ hence the volume of E is given by $V = M/\delta = \frac{2\pi}{3}\left(1-\frac{1}{\sqrt{2}}\right)$. If δ is not constant then the volume and mass are not proportional in general. That said, let us continue to the calculation of the coordinates of the center of mass. Begin by using symmetry to see that $x_{cm} = y_{cm} = 0$ hence all we need to explicitly calculate is z_{cm}.

$$\begin{aligned} z_{cm} = \frac{1}{M}\iiint_E z\delta dV &= \frac{\delta}{M}\int_0^{2\pi}\int_0^1\int_0^{\pi/4} (\rho\cos\phi)(\rho^2\sin\phi\, d\phi\, d\rho\, d\theta) \\ &= \frac{\delta}{M}\theta\Big|_0^{2\pi}\left(\frac{1}{4}\rho^4\Big|_0^1\right)\left(\frac{1}{2}\sin^2\phi\Big|_0^{\pi/4}\right) \\ &= \frac{\delta}{M}(2\pi)\left(\frac{1}{4}\right)\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{\delta}{M}\frac{\pi}{4}. \end{aligned}$$

$\left(0, 0, \frac{\pi\delta}{4M}\right)$

[10] I often use ρ for density in physics, but, it seems that would be a bad notation here

Thus,

$$z_{cm} = \frac{\pi}{4} \cdot \left[\frac{2\pi}{3}\left(1 - \frac{1}{\sqrt{2}}\right)\right]^{-1} = \frac{\pi}{4} \cdot \frac{3}{2\pi} \cdot \frac{\sqrt{2}}{\sqrt{2}-1} = \frac{3\sqrt{2}}{8(\sqrt{2}-1)}.$$

In summary, the center of mass is at $(0, 0, \frac{3\sqrt{2}}{8(\sqrt{2}-1)})$.

Sometimes we are interested in the **geometric center** or **centroid** of an object. This can be found in the same manner as the preceding example. We simply set $\delta = 1$ and find the center of mass. This is merely a convenience, if we set δ to any nonzero constant then the resulting center of mass would still be the same. Only in the case that the mass density is variable do we find a possible distinction between the **centroid** and **center of mass**.

Example 6.6.12. *Find the kinetic energy of a ball with radius R and mass m that spins with angular velocity ω and moves linearly with speed v.*

Solution: *It is shown in physics that $KE_{net} = KE_{rot} + KE_{trans}$. Where $KE_{trans} = \frac{1}{2}mv^2$ and $KE_{rot} = \frac{1}{2}I\omega^2$ where I is the moment of inertia measured with respect to the axis of rotation. We take the (moving) z-axis to be the rotation axis. From physics,*

$$\begin{aligned}
I_z &= \iint_E \delta(x,y,z)(x^2+y^2)dV \qquad \textit{constand density has } \delta = \frac{m}{\frac{3}{4}\pi R^3} \\
&= \int_0^{2\pi}\int_0^{\pi}\int_0^R \frac{3m}{4\pi R^3}\left[(\rho\cos\theta\sin\phi)^2 + (\rho\sin\theta\sin\phi)^2\right]\rho^2\sin\phi\, d\rho\, d\phi\, d\theta \\
&= \frac{3m}{4\pi R^3}\int_0^{2\pi} d\theta \int_0^{\pi}\int_0^R \rho^4 \sin^3\phi\, d\rho\, d\phi \qquad : \int \sin^3\phi d\phi = \int (1-\cos^2\phi)\sin\phi d\phi \\
&= \frac{3m(2\pi)}{4\pi R^3}\left(\frac{1}{3}\cos^3\phi - \cos\phi\Big|_0^{\pi}\left(\frac{\rho^5}{5}\Big|_0^R\right)\right) \\
&= \frac{3m}{4\pi R^3}(2\pi)\left(\frac{-1}{3} + 1 - \frac{1}{3} + \cos(0)\right)\left(\frac{R^5}{5}\right) \\
&= \frac{3m(2\pi)}{4\pi R^3}\cdot\frac{4}{3}\cdot\frac{R^5}{5} \\
&= \boxed{\frac{2}{5}mR^2 = I_{sphere}}
\end{aligned}$$

Hence, we find:

$$KE_{total} = \frac{1}{2}mv^2 + \frac{1}{5}mR^2\omega^2$$

If the ball is rolling without slipping then $\omega = v/R$

$$KE_{total} = \frac{1}{2}mR^2\omega^2 + \frac{1}{5}mR^2\omega^2 = \frac{7}{10}mR^2\omega^2.$$

The formulas for center of mass or moment of inertia for a given solid all required multivariate integration. In physics, I usually begin by describing the concepts for a finite set of particles then we imagine how as the number of particles increases we can pass to the continuum. In that smearing process the finite sums elevate to continuous sums which we know and love as integrals. I forego this argument from the finite to the continuum as the content is mostly physical. In this course, in these notes, I merely supply some formulas given to us by the subject of physics and we use them as interesting problems to hone our integration skill.

Example 6.6.13. *Let $E = \{(x, y, z) \mid x^2/a^2 + y^2/b^2 + z^2/c^2 \leq 1\}$ calculate the volume of E.*

Solution: *by making the change of coordintes $x = au$, $y = bv$ and $z = cw$. Under this change of coordinates E morphs from an ellipsoid to a sphere B;*

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = (au)^2/a^2 + (bv)^2/b^2 + (cw)^2/c^2 = u^2 + v^2 + w^2$$

Let $B = \{(u, v, w) | u^2 + v^2 + w^2 \leq 1\}$ and calculate:

$$\begin{aligned}
\iiint_E dV &= \iiint_E dx\,dy\,dz \\
&= \iiint_B \frac{\partial(x,y,z)}{\partial(u,v,w)} du\,dv\,dw \qquad : \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc \\
&= \iiint_B abc\,du\,dv\,dw \\
&= abc \underbrace{\iiint_B du\,dv\,dw}_{\text{volume of sphere } = \frac{4}{3}\pi} \\
&= \boxed{\frac{4\pi abc}{3}}
\end{aligned}$$

$$\begin{aligned}
\text{Vol Sphere} &= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta \\
&= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi} \sin\phi\, d\phi\right)\left(\int_0^R \rho^2 d\rho\right) \\
&= (2\pi)(2)(R^3/3) \\
&= \boxed{\tfrac{4}{3}\pi R^3}
\end{aligned}$$

The volume of the unit sphere is calculated as follows,

$$\iiint_B du\,dv\,dw = \int_0^1 \int_0^{\pi} \int_0^{2\pi} \left| \frac{\partial(u,v,w)}{\partial(\rho,\theta,\phi)} \right| d\theta\, d\phi\, d\rho \qquad : \begin{cases} u = \rho\cos\theta\sin\phi \\ v = \rho\sin\theta\sin\phi \\ w = \rho\cos\phi \end{cases}$$

Calculate the Jacobian,

$$\begin{aligned}
\frac{\partial(u,v,w)}{\partial(\rho,\theta,\phi)} &= \begin{vmatrix} u_\rho & u_\theta & u_\phi \\ v_\rho & v_\theta & v_\phi \\ w_\rho & w_\theta & w_\phi \end{vmatrix} = \begin{vmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix} \\
&= -\rho^2\cos^2\theta\sin^3\phi + \rho^2\sin\theta\sin\phi[-\sin^2\phi\sin\theta - \cos^2\phi\sin\theta] - \rho^2\cos^2\theta\cos^2\phi\sin\phi \\
&= -\rho^2\cos^2\theta\sin\phi - \rho^2\sin^2\theta\sin\phi \\
&= -\rho^2\sin\phi
\end{aligned}$$

Thus,

$$\iiint_B du\,dv\,dw = \int_0^1 \int_0^{\pi} \int_0^{2\pi} \rho^2 \sin\phi d\theta\, d\rho = 2\pi \left(\frac{\rho^3}{3}\Big|_0^1 \right) \left(-\cos\phi \Big|_0^{\pi} \right) = \frac{4\pi}{3}.$$

In general, the detail given at the end of the previous example is not required. The volume of a sphere is known for future reference. However, you ought to be able to work out the details.

Example 6.6.14. *Let E be the finite cylinder defined by $x^2+y^2 \leq 16$ and $-5 \leq z \leq 4$. Calculate $\iiint_E \sqrt{x^2+y^2}\, dV$.*

Solution: *observe E is $0 \leq r \leq 4$ and $-5 \leq z \leq 4$. Moreover, as there is no restriction on θ we have $0 \leq \theta \leq 2\pi$ for E.*

$$\begin{aligned}
\iiint_E \sqrt{x^2+y^2}\, dV &= \int_0^{2\pi}\int_{-5}^{4}\int_0^4 r^2 dr\, dz\, d\theta && : \text{ recall } dV = r dr\, dz\, d\theta \\
&= \int_0^{2\pi} d\theta \int_{-5}^{4} dz \int_0^4 r^2 dr && : \text{ since } \left|\frac{\partial(x,y,z)}{\partial(r,z,\theta)}\right| = r \\
&= (2\pi)(9)(16/3) \\
&= \boxed{\frac{384\pi}{3}}
\end{aligned}$$

Example 6.6.15. *Let $E = \{(x,y,z) \mid 0 \leq z \leq x+y+5,\ 4 \leq x^2+y^2 \leq 9\}$. Calculate $\iiint_E x\, dV$*

Solution: *We can convert E to cylindrical coordinates via $x = r\cos\theta, y = r\sin\theta$ and of course $z = z$. Observe $x^2+y^2 = r^2$, E becomes:*

$$\begin{aligned}
&0 \leq \theta \leq 2\pi \\
&4 \leq r^2 \leq 9 \Rightarrow 2 \leq r \leq 3 \\
&0 \leq z \leq r\cos\theta + r\sin\theta + 5.
\end{aligned}$$

Thus motivating,

$$\begin{aligned}
\iiint_E x\, dV &= \int_0^{2\pi}\int_2^3\int_0^{r(\cos\theta+\sin\theta)+5} (r^2\cos\theta) dz\, dr\, d\theta, \qquad dV = r\, dz\, dr\, d\theta \\
&= \int_0^{2\pi}\int_2^3 \left(z\Big|_0^{r(\cos\theta+\sin\theta)+5} \right) r^2\cos\theta\, dr\, d\theta \\
&= \int_0^{2\pi}\int_2^3 \left[r^3(\cos^2\theta + \sin\theta\cos\theta) + 5r^2\cos\theta \right] dr\, d\theta \\
&= \int_0^{2\pi} \left[\left(\frac{r^4}{4}\Big|_2^3 \right)(\cos^2\theta + \sin\theta\cos\theta) + \left(\frac{5}{3}r^3\Big|_2^3 \cos\theta \right) \right] d\theta \\
&= \int_0^{2\pi} \left[\frac{65}{4}\left(\frac{1}{2} + \frac{1}{2}\cos(2\theta) + \frac{1}{2}\sin(2\theta) \right) + \frac{95}{3}\cos\theta \right] d\theta \qquad \star \\
&= \int_0^{2\pi} \frac{65}{8} d\theta \\
&= \frac{65}{8}\theta \Big|_0^{2\pi} \\
&= \boxed{\frac{65\pi}{4}}
\end{aligned}$$

In the $\star$ step I observe all terms, except the first constant term, integrate away due as a sinusoidal wave-form has equal areas above and below the x-axis during any integer multiple of periods. This observation can be very labor saving in problems such as this.

Example 6.6.16. *Find mass M of a ball B which is the set of all points (x, y, z) such that $x^2 + y^2 + z^2 \le a^2$ where a is some constant and the mass-density at (x, y, z) is given by:*

$$\rho = k\sqrt{x^2+y^2} = kr \qquad \textit{(proportionality constant is } k\textit{)}$$

Note $x^2 + y^2 + z^2 \le a^2$ implies $r^2 + z^2 \le a^2$ hence $z^2 \le a^2 - r^2$ and consequently the sphere $\rho \le a$ in cylindrical coordinates is:

$$0 \le \theta \le 2\pi, \qquad 0 \le r \le a, \qquad -\sqrt{a^2-r^2} \le z \le \sqrt{a^2-r^2}.$$

description of sphere in cylindrical coord.

Recall $\rho = \frac{dm}{dV}$ thus $dm = \rho dV$. Therefore, we may calculate M by integrating ρdV:

$$\begin{aligned} M = \int dm &= \iiint_B \rho dV \\ &= \int_0^{2\pi}\int_0^a \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} kr^2 dz\,dr\,d\theta \\ &= \int_0^a 2\pi k r^2 \left(z\Big|_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} \right) dr \qquad : \textit{ the } \theta \textit{ integral yields } 2\pi \\ &= 4\pi k \int_0^a \underbrace{r^2\sqrt{a^2-r^2}dr}_{\textit{not an obvious integral}} \end{aligned}$$

We can make a $r = a\sin b$ substitution then $a^2 - r^2 = a^2(1-\sin^2 b) = a^2\cos^2 b$. Moreover, $dr = a\cos b\, db$ hence $r^2\sqrt{a^2-r^2}dr = a^4\sin^2 b\cos^2 b\, db$. I like to use imaginary exponentials to work out the trigonometry here. Note $\sin b = \frac{1}{2i}(e^{ib} - e^{-ib})$ and $\cos b = \frac{1}{2}(e^{ib} + e^{-ib})$ hence

$$\begin{aligned} \sin^2 b\cos^2 b &= \left(\frac{1}{2}\right)^2\left(\frac{1}{2i}\right)^2 (e^{ib} - e^{-ib})^2(e^{ib}+e^{-ib})^2 \\ &= \frac{-1}{16}(e^{2ib} - 2 + e^{-2ib})(e^{2ib} + 2 + e^{-2ib}) \\ &= \frac{-1}{16}(e^{4ib} + 2e^{2ib} + 1 + 2e^{2ib} - 4 + 2e^{-2ib} + 1 - 2e^{-2ib} + e^{-4ib}) \\ &= -\frac{1}{8}\frac{1}{2}(e^{4ib} + e^{-4ib}) + \frac{1}{8} \\ &= \frac{1}{8}(1 - \cos(4b)) \end{aligned}$$

Furthermore, note $r = a$ gives $b = \frac{\pi}{2}$ whereas $r = 0$ yields $b = 0$. Therefore,

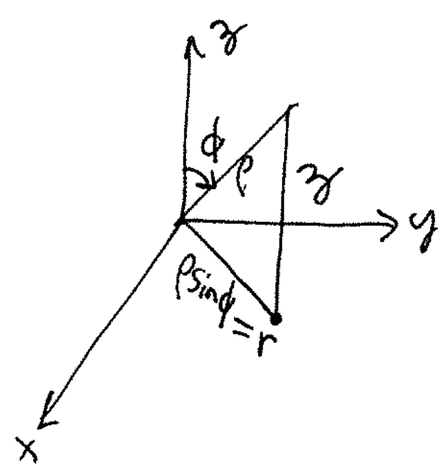
$$M = 4\pi k \int_0^{\frac{\pi}{2}} \frac{a^4}{8}(1-\cos(4b))db = \frac{\pi k a^4}{2}\cdot\frac{\pi}{2} = \boxed{\frac{k\pi^2 a^4}{4}}.$$

Naturally, you might use a table of integrals or a computer algebra system to tackle the integral in the precious example if you did not know how to calculate it from base principles. It is often the case that a given integral is easier if you **choose natural bounds**. The last example had a integrand which was manifestly cylindrical and bounds which were simplest in spherical coordinates. The example below shows how the integration simplifies if we change the integrand to sphericals rather than changing the bounds to cylindricals as in the preceding example.

Example 6.6.17. *Calculate M from Example 6.6.16 via spherical coordinates: note that*[11] $r = \rho\sin\phi$ *connects the spherical radius ρ and the cylindrical radius r.*

$\delta\cancel{\phi} = kr = k\rho\sin\phi$

$$\begin{aligned} m &= \iiint_B \rho dV \\ &= \int_0^{\pi}\int_0^{2\pi}\int_0^{a} k\rho\sin\phi\rho^2\sin\phi\, d\rho\, d\theta\, d\phi \\ &= k\int_0^{\pi}\sin^2\phi\, d\phi\int_0^{2\pi}d\theta\int_0^{a}\rho^3 d\rho \\ &= k\cdot\frac{\pi}{2}\cdot(2\pi)\cdot\frac{a^4}{4} \\ &= \boxed{\frac{k\pi^2a^4}{4}}. \end{aligned}$$



Moral of story: **coordinate choice matters.**

Example 6.6.18. *Let $B : (x, y, z)$ with $x^2 + y^2 + z^2 \leq 25$. Calculate $\iiint_B (x^2+y^2+z^2)^2 dV$.*

Solution: *Recall $dV = \rho^2\sin\phi d\theta\, d\phi\, d\rho$ and note B is clearly $\rho \leq 5$ in spherical coordinates where θ and ϕ are free to range over their entire domains:*

$$\begin{aligned} \iiint_B (x^2+y^2+z^2)^2 dV &= \int_0^{5}\int_0^{\pi}\int_0^{2\pi}\rho^6\sin\phi\, d\theta\, d\phi\, d\rho \\ &= \int_0^{2\pi}d\theta\int_0^{\pi}\sin\phi\, d\phi\int_0^{5}\rho^6 d\rho \\ &= (2\pi)(2)\left(\frac{5^7}{7}\right) = \boxed{\frac{312,500\pi}{7}}. \end{aligned}$$

The integration of the above example is a fairly typical of integration over spheres. When the integrand has no angular dependence the θ and ϕ integrations produce a factor of 4π in total over a spherical region. The **solid angle** Ω measures both the polar and azmuthial angular displacements. In particular, $d\Omega = \sin\phi d\phi d\theta$ and over a sphere we find the total solid angle is 4π.

Example 6.6.19. *Let E be the region described in spherical coordinates by $1 \leq \rho \leq 2$ and $0 \leq \theta \leq \frac{\pi}{2}$ where $0 \leq \phi \leq \frac{\pi}{2}$. Calculate $\iiint_E z dV$.*

[11]geometrically obvious, algebraically follows from $r^2 = x^2 + y^2 = \rho^2\cos^2\theta\sin^2\phi + \rho^2\sin^2\theta\sin^2\phi = \rho^2\sin^2\phi$.

Solution: *we use spherical coordinates to calculate the integral.*

$$\begin{aligned}
\iiint_E z dV &= \int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 (\rho\cos\phi)(\rho^2\sin\phi\, d\rho\, d\phi\, d\theta) \\
&= \int_0^{\pi/2} d\theta \int_0^{\pi/2} \sin\phi\cos\phi d\phi \int_1^2 \rho^3 d\rho \\
&= \left(\theta\Big|_0^{\pi/2}\right)\left(\frac{1}{2}\sin^2\phi\Big|_0^{\pi/2}\right)\left(\frac{1}{4}\rho^4\Big|_1^2\right) \\
&= \left(\frac{\pi}{2}\right)\left(\frac{1}{2}\right)\left(\frac{16-1}{4}\right) \\
&= \boxed{\frac{15\pi}{16}}
\end{aligned}$$

Example 6.6.20. *Find volume of part of ball $\rho \leq a$ that lies between $\phi = \pi/6$ and $\phi = \pi/3$.*

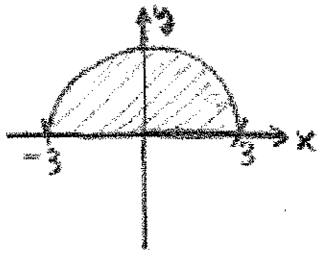
$$\begin{aligned}
V = \underbrace{\iiint dV}_{\text{part of ball}} &= \int_0^a \int_0^{2\pi}\int_{\pi/6}^{\pi/3} \rho^2 \sin\phi\, d\phi\, d\theta\, d\rho \\
&= \int_0^a \rho^2 d\rho \int_0^{2\pi} d\theta \int_{\pi/6}^{\pi/3} \sin\phi d\phi \\
&= \left(\frac{1}{3}\rho^3\Big|_0^a\right)\left(\theta\Big|_0^{2\pi}\right)\left(-\cos\phi\Big|_{\pi/6}^{\pi/3}\right) \\
&= \left(\frac{1}{3}a^3\right)(2\pi)\left(-\cos(\frac{\pi}{3}) + \cos(\frac{\pi}{6})\right) \\
&= \frac{2\pi a^3}{3}\left(\frac{-1}{2} + \frac{\sqrt{3}}{2}\right) \\
&= \boxed{\frac{\pi a^3}{3}(\sqrt{3}-1)}
\end{aligned}$$

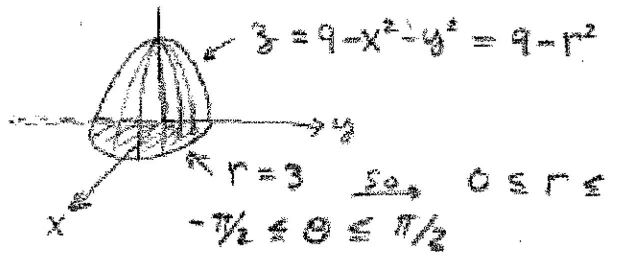
Example 6.6.21. *Let E be the region with $0 \leq \rho \leq 1$ and $0 \leq \phi \leq \pi/2$ with $0 \leq \theta \leq 2\pi$,*

$$\begin{aligned}
\iiint_E (x^2+y^2)dV &= \int_0^1\int_0^{\pi/2}\int_0^{2\pi} (\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta)\rho^2\sin\phi d\theta d\phi d\rho \\
&= \int_0^1\int_0^{\pi/2}\int_0^{2\pi} \rho^4\sin^3\phi d\theta d\phi d\rho \\
&= \int_0^1 \rho^4 d\rho \int_0^{2\pi} d\theta \int_0^{\pi/2} (1-\cos^2\phi)\sin\phi d\phi \\
&= \frac{1}{5}\cdot 2\pi \cdot \left(\frac{1}{3}\cos^3\phi - \cos\phi\right)\Big|_0^{\pi/2} \\
&= \boxed{\frac{4\pi}{15}}
\end{aligned}$$

Example 6.6.22. *Let* $I = \int_{-3}^{3}\int_0^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2} \sqrt{x^2+y^2}\,dz\,dy\,dx$. *Calculate* I *by changing to cylindrical coordinates.*

Solution: *notice the integration region has* $-3 \leq x \leq 3$ *and* $0 \leq y \leq \sqrt{9-x^2}$ *whereas* $0 \leq z \leq 9 - x^2 - y^2$. *Thus:*





Hence integrate, notice $dV = r\,dr\,d\theta\,dz$ *whose product with the integrand of* r *yields* r^2,

$$\begin{aligned}
I &= \int_{-\pi/2}^{\pi/2}\int_0^3\int_0^{9-r^2} r^2 dz dr d\theta \qquad : used\, dV = r dr\, d\theta\, dz \text{ and } \sqrt{x^2+y^2} = \sqrt{r^2} = r \\
&= \int_{-\pi/2}^{\pi/2}\int_0^3 (9r^2 - r^4) dr d\theta \\
&= \int_{-\pi/2}^{\pi/2} \left(3r^3 - \frac{1}{5}r^5\right)\Big|_0^3 d\theta \\
&= \pi\left(81 - \frac{1}{5}(243)\right) = \left(\frac{405-243}{5}\right)\pi = \boxed{\frac{162\pi}{5}}
\end{aligned}$$

6.7 integration in n variables

Let $\mathbb{R}^n$ have Cartesian Coordinates$(x_1, x_2, \cdots, x_n)$ and suppose $T(x_1, x_2, \cdots, x_n) = (y_1, y_2, \cdots, y_n)$ where y_i is a function of $x_1, x_2, \cdots, x_n$. For each i and we suppose T has DT invertible over the domain of integration below,

$$\iint_S \cdots \int f(x_1, x_2, \cdots, x_n) d^n x = \iint_{T(S)} \cdots \int f(y_1, y_2, \cdots, y_n)|det(DT)| d^n y$$

where $d^n x \equiv dx_1 dx_2 \cdots dx_n$ and $d^n y = dy_1 dy_2 \cdots dy_n$. The meaning of the n-fold integration should be an easy generalization of $n = 2$ and $n = 3$ which we have already treated in some depth.

Example 6.7.1. *The Hypersphere:* $(x, y, z, w) \in \mathbb{R}^4$ *such that* $x^2+y^2+z^2+w^2 \leq R^2$. *Generalized Spherical Coordinates are*

$$\begin{aligned}
x &= r\cos\theta\sin\phi\sin\psi \\
y &= r\sin\theta\sin\phi\sin\psi \\
z &= r\cos\phi\sin\psi \\
w &= r\cos\psi
\end{aligned}$$

Where $0 \leq \theta \leq 2\pi$ *and* $0 \leq \phi, \psi \leq \pi$. *You can check that* $x^2 + y^2 + z^2 + w^2 = r^2$. *Then it's a long but straightforward calculation,* $\left|\frac{\partial(x,y,z,w)}{\partial(r,\theta,\phi,\psi)}\right| = r^3 \sin^2\psi \sin\phi$ *And consequently if we integrate* $d^4x = dx\,dy\,dz\,dw$ *in generalized spherical coordinates then we find:*

$$Vol_4(Hypersphere) = \int_0^R \int_0^{2\pi} \int_0^{\pi} \int_0^{\pi} r^3 \sin^2\psi \, \sin\phi \, d\psi \, d\phi \, d\theta \, dr = \frac{\pi^2 R^4}{2}.$$

I used this result to help calculate Guass' Law in 4 spatial dimensions in my Math 430 notes.

Example 6.7.2. *Find the volume bounded by* $x^2 + y^2 = R^2$ *and* $0 \leq z, w \leq h$. *This gives a finite hypercylinder. We calculate* $d^4x = dx\,dy\,dz\,dw = r\,dr\,d\theta\,dz\,dw$ *by introducing cylindrical coordinates* $x = r\cos\theta$ *and* $y = r\sin\theta$ *whereas* $z = z$ *and* $w = w$. *Thus, the hypervolume of the generalized cylinder is calculated by the following integral:*

$$\begin{aligned} V_4 &= \int_0^h \int_0^h \int_0^{2\pi} \int_0^R r\,dr\,d\theta\,dz\,dw \\ &= \int_0^h dw \int_0^h dz \int_0^{2\pi} d\theta \int_0^R r\,dr \\ &= \pi h^2 R^2. \end{aligned}$$

Example 6.7.3. *Consider* $E = \{(x, y, z, t) \mid x^2 + y^2 + z^2 \leq R^2, \ 0 \leq t \leq h\}$. *This hypervolume consists of a solid sphere of radius* R *attached at each* t *along* $[0, h]$. *The hypervolume is found by changing the* x, y, z *to spherical coordinates where are earlier work still applies:*

$$\begin{aligned} V_4 &= \int\int\int\int_E d^4x \\ &= \int_0^h \int_0^{2\pi} \int_0^{\pi} \int_0^R \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta dt \\ &= \int_0^h dt \int_0^{2\pi} \int_0^{\pi} \int_0^R \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\ &= \frac{4\pi h R^3}{3}. \end{aligned}$$

You might find my examples here a bit contrived. However, if application to the real world is what you seek then be assured that integrals over spaces with more than three variables are found in physics. In particular, if you study statistical mechanics of n-particles then you will face such integrals.