LECTURE 28: ALGEBRA & GEOMETRY OF VOLUME ELEMENT

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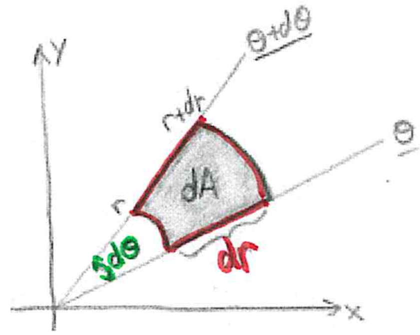
6.8 algebra and geometry of area and volume elements

In this section we take another look at how we can understand dA and dV. I present two methods which complement the Jacobian-based derivation we have thus far considered. Geometrically, we consider infinitesimal area elements or volume elements and direct geometric investigation yields the volume which correspond to small changes in the curvelinear coordinates. Algebraically, we introduce the **wedge product** and show by example how it also allows us to calculate the Jacobian factor. We do not attempt a general treatment here so we investigate only polar, cylindrical and spherical coordinates.

Let us begin with the two-dimensional case. For polar coordinates, we can derive $dA = r\,dr\,d\theta$ by examining the area of the sector pictured below:



$dl_r = dr$

$dl_\theta = r d\theta$

$dA = dl_r dl_\theta = r dr d\theta$

The region above is an infinitesimal **polar rectangle** (not to scale !). Formally,

$$dA = \frac{1}{2}(r+dr)^2 d\theta - \frac{1}{2}r^2 d\theta = \frac{1}{2}(r^2 + 2rdr + (dr)^2 - r^2)d\theta = \boxed{r\,dr\,d\theta = dA}$$

We neglect the term $\frac{1}{2}dr^2 d\theta$ as it is much smaller than the other terms as we consider $dr, d\theta \ll 1$. Of course, this notation is just a formalism which we use in place of a careful argument involving finite differences. You'll find the finite argument and how it passes to an integral in any number of standard texts. Here, we embrace the infinitesimal method and forge ahead.

$df = f_x dx + f_y dy$

The algebraic method to derive these is given by differential forms and the wedge product. The basic rules are that d on a function gives the total differential and the **wedge product** is denoted $\wedge$ which is an associative product with the usual algebraic rules except, it is not commutative in general. For differentials, $df \wedge dg = -dg \wedge df$. In particular, this means $dx \wedge dx = 0$ and $dy \wedge dy = 0$ as well as $dr \wedge dr = 0$ and $d\theta \wedge d\theta = 0$. Let us proceed with these basic ideas in mind. We use $x = r\cos\theta$ and $y = r\sin\theta$ in the second equality:

$dA = dxdy = dydx$

$$\begin{aligned}
d\vec{A} &= dx \wedge dy \\
&= d(r\cos\theta) \wedge d(r\sin\theta) \\
&= [\cos\theta dr - r\sin\theta d\theta] \wedge [\sin\theta dr + r\cos\theta d\theta] \\
&= \sin\theta\cos\theta dr \wedge dr + r\cos^2\theta dr \wedge d\theta - r\sin^2\theta d\theta \wedge dr - r^2\sin\theta\cos\theta d\theta \wedge d\theta \\
&= r\cos^2\theta dr \wedge d\theta + r\sin^2\theta dr \wedge d\theta \\
&= \boxed{rdr \wedge d\theta = d\vec{A}.}
\end{aligned}$$

0 $-dr \wedge d\theta$ 0 $= -r d\theta \wedge dr$

Here $d\vec{A} = -r\,d\theta \wedge dr$ also, in contrast to our usual $dA = r\,dr\,d\theta = r\,d\theta\,dr$ the minus sign encodes the **orientation**. The distinction between $dx \wedge dy$ and $dx\,dy$ is important. The scalar quantity

$dx\,dy$ has no sense of up or down whereas the fact that $dx \wedge dy = -dy \wedge dx$ allows us to identify $dx \wedge dy$ as the area element for the upwards oriented xy-plane. A little later in this course we discuss the **vector area element**. Essentially my comment here is that $dx \wedge dy$ is naturally identified with the vector area element of the plane oriented with $\widehat{z}$.

Wedge products and determinants are closely related. In particular:

$$\boxed{A\widehat{x}_1 \wedge A\widehat{x}_2 \wedge \cdots \wedge A\widehat{x}_n = \det(A)\,\widehat{x}_1 \wedge \widehat{x}_2 \wedge \cdots \wedge \widehat{x}_n.}$$

The identity above allows us to calculate determinants implicitly from wedge products. Some texts take this as the definition of the determinant of a matrix. Let us work out the 2×2 case. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then observe:

$$A\widehat{x}_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} = a\,\widehat{x}_1 + c\,\widehat{x}_2.$$

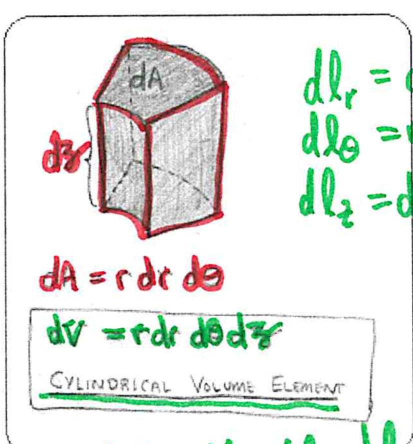
$$A\widehat{x}_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix} = b\,\widehat{x}_1 + d\,\widehat{x}_2.$$

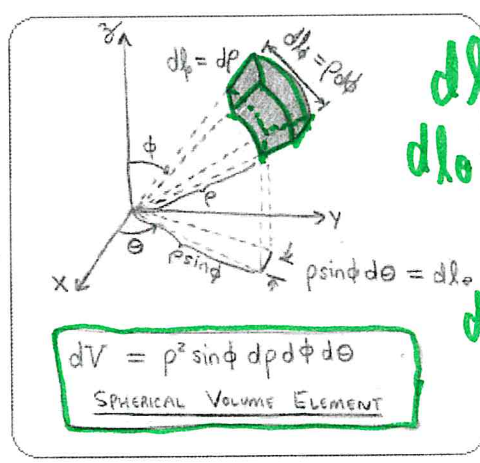
Therefore,

$$\begin{aligned} A\widehat{x}_1 \wedge A\widehat{x}_2 &= (a\,\widehat{x}_1 + c\,\widehat{x}_2) \wedge (b\,\widehat{x}_1 + d\,\widehat{x}_2) \\ &= ac\,\widehat{x}_1 \wedge \widehat{x}_1 + ad\,\widehat{x}_1 \wedge \widehat{x}_2 + cb\,\widehat{x}_2 \wedge \widehat{x}_1 + cd\,\widehat{x}_2 \wedge \widehat{x}_2 \\ &= (ad - cb)\,\widehat{x}_1 \wedge \widehat{x}_2 \end{aligned}$$

Thus, $(ad - cb)\,\widehat{x}_1 \wedge \widehat{x}_2 = \det(A)\,\widehat{x}_1 \wedge \widehat{x}_2$ thus comparing the coefficient of $\widehat{x}_1 \wedge \widehat{x}_2$ we find $\boxed{\det(A) = ad - cb}$. This is the usual formula for the 2×2 determinant. There is much more to say about the algebraic significance of the wedge product beyond this course. We simply introduce the reader to some of the basic benefits of the $\wedge$-product.

We now turn to the three dimensional case. I attempt to illustrate typical infinitesimal solid regions for cylindrical and spherical coordinates below:





In each case, the infinitesimal volume is obtained by multiplying the coordinate arclengths.

The algebraic method for volume elements follows the natural pattern we already saw for area elements. Again the wedge product yields something a bit different than a simple dV. I will denote the object calculated by $d\vec{V}$ to emphasize there is a distinction. For example, $dV = dx\,dy\,dz = dy\,dx\,dz$ and the distinction is merely the indicated order of an iterated integration. In contrast, by definition, $d\vec{V} = dx \wedge dy \wedge dz = -dy \wedge dx \wedge dz$ and the sign indicates that $dx \wedge dy \wedge dz$ orients a volume in an opposite sense to $dy \wedge dx \wedge dz$. If we could visualize four dimensions (x, y, z, w), then we could envision $dx \wedge dy \wedge dz$ as giving the hyperplane $w = k$ an upward orientation whereas $dy \wedge dx \wedge dz$ gives the hyperplane $w = k$ a downward-pointing orientation. In any event, we will think more on orientation when we study line and surface integrals which are both defined with respect to oriented objects. That said, I hope you can appreciate the following calculations. They are a large part of what sparked my initial interest in the topic of **differential forms**.

The cylindrical volume form follows from the work we already did to calculate the polar area form:

$$d\vec{V} = dx \wedge dy \wedge dz = d(r\cos\theta) \wedge d(r\sin\theta) \wedge dz = \boxed{r dr \wedge d\theta \wedge dz.}$$

In contrast, the spherical volume element requires some effort[12]:

$$\begin{aligned}
d\vec{V} &= dx \wedge dy \wedge dz \\
&= d(\rho\sin\phi\cos\theta) \wedge d(\rho\sin\phi\sin\theta) \wedge d(\rho\cos\phi) \\
&= [\sin\phi\cos\theta d\rho + \rho\cos\phi\cos\theta d\phi - \rho\sin\phi\sin\theta d\theta] \\
&\qquad \wedge [\sin\phi\sin\theta d\rho + \rho\cos\phi\sin\theta d\phi + \rho\sin\phi\cos\theta d\theta] \\
&\qquad \wedge [\cos\phi d\rho - \rho\sin\phi d\phi] \\
&= [\sin\phi\cos\theta d\rho + \rho\cos\phi\cos\theta d\phi - \rho\sin\phi\sin\theta d\theta]\wedge \\
&\qquad [\rho\sin\theta d\phi \wedge d\rho + \rho^2\sin\phi\cos\phi\cos\theta d\theta \wedge d\rho - \rho^2\sin^2\phi\cos\theta d\theta \wedge d\phi] \\
&= -\rho^2\sin^3\phi\cos^2\theta d\rho \wedge d\theta \wedge d\phi + \rho^2\cos^2\phi\sin\phi\cos^2\theta d\phi \wedge d\theta \wedge d\rho + \\
&\qquad \rho^2\sin^3\phi\sin^2\theta d\theta \wedge d\rho \wedge d\phi - \rho^2\sin\phi\cos^2\phi\sin^2\theta d\theta \wedge d\phi \wedge d\rho \\
&= [-\rho^2\sin^3\phi\cos^2\theta - \rho^2\cos^2\phi\sin\phi\cos^2\theta - \rho^2\sin^3\phi\sin^2\theta - \rho^2\sin\phi\cos^2\phi\sin^2\theta]d\rho \wedge d\theta \wedge d\phi \\
&= \rho^2\sin\phi[\sin^2\phi\cos^2\theta + \cos^2\theta\cos^2\phi + \sin^2\phi\sin^2\theta + \sin^2\theta\cos^2\theta]d\rho \wedge d\phi \wedge d\theta \\
&= \rho^2\sin\phi[\cos^2\theta(\sin^2\phi + \cos^2\phi) + \sin^2\theta(\sin^2\phi + \cos^2\phi)]d\rho \wedge d\phi \wedge d\theta \\
&= \boxed{\rho^2\sin\phi d\rho \wedge d\phi \wedge d\theta}
\end{aligned}$$

If you would like to learn more about differential forms then you might read my Advanced Calculus notes. In my current formulation, I spend about half the course developing and applying differential forms. A treatment with just calculus III in mind can be found in Susan Colley's *Vector Calculus*. She devotes a whole chapter to the exposition of basic differential forms.

6.9 Problems

Problem 151 Calculate

$$\int_0^2 \int_0^4 (3x+4y)\, dx\, dy$$

[12]you could try to follow this line by line, but, if you do attempt it, it is better to try your own path

Problem 152

$$\int_0^{\pi/2}\int_0^{\pi/2} \sin(x)\cos(y)\, dx\, dy$$

Problem 153

$$\int_{-1}^{1}\int_0^{1} \sin^3(x)\cos^{42}(y)\, dy\, dx$$

Problem 154 Calculate the average of $f(x, y) = x^2 + y^2$ on the unit-square.

Problem 155 Calculate the average of $f(x, y) = x^2 + y^2$ on the region bounded by $x^2 + y^2 = R^2$.

Problem 156 Calculate the average of $f(x, y) = xy$ on $[1, 2] \times [3, 4]$.

Problem 157 Show that

$$\lim_{n\to\infty}\int_0^1\int_0^1 x^n\, y^n\, dx\, dy = 0.$$

Problem 158 Calculate

$$\int_0^{\ln(2)}\int_0^{\ln(3)} e^{x+y} dx\, dy$$

Problem 159 Suppose $\int\int_R f\, dA = \int_0^1 \int_{x^2}^{\sqrt{x}} (1 + x) dy\, dx$. Calculate the given integral.

Problem 160 For the integral given in the previous problem, explicitly write R as a subset of $\mathbb{R}^2$ using set-builder notation. In addition, calculate the integral once more with the interation of the integrals beginning with dx. Draw a picture to explain the inequalities which form the basis for your new set-up to the integral.

Problem 161 Reverse the order of integration in order to calculate the following integral:

$$\int_0^{\sqrt{\pi}}\int_y^{\sqrt{\pi}} \sin(x^2)\, dx\, dy.$$

Problem 162 Reverse the order of integration in order to calculate the following integral:

$$\int_0^1\int_y^1 \frac{1}{1+x^4}\, dx\, dy.$$

Problem 163 Find the average of $f(x, y) = xy$ over the triangle with vertices $(0, 0), (3, 1)$ and $(-2, 4)$.

Problem 164 Find volume bounded by $z = y + e^x$ and the xy-plane for $(x, y) \in [0, 1] \times [0, 2]$.

Problem 165 Find the volume bounded inside the cylinder $x^2 + y^2 = 1$ and the planes $z = x + 1$ and $z = y - 3$.

Problem 166 Find the volume bounded by the coordinate planes and the plane $3x + 2y + z = 6$.

Problem 167 Calculate the integral (use polars):

$$\int_0^2 \int_0^{\sqrt{4-x^2}} (x^2+y^2)^{3/2}\, dy\, dx.$$

Problem 168 Calculate the integral (use polars):

$$\int_0^1 \int_x^1 (x^2+y^2)^3\, dy\, dx.$$

Problem 169 Suppose R is the region bounded by $y + |x|$ and $x^2 + (y-1)^2 = 1$. Express R in polar coordinates. In other words, draw a picture and indicate how the points in R are reached by particular ranges of r and θ.

Problem 170 Find volume bounded by the paraboloid $x = y^2 + 2z^2$ and the parabolic cylinder $x = 2 - y^2$.

Problem 171 Find the volume bounded by the cylinder $x^2 + y^2 = 1$ and $z = 2 + x + y$ and $z = 1$.

Problem 171 Find the volume bounded by the cones $z = \sqrt{x^2+y^2}$ and $z = 2\sqrt{x^2+y^2}$ and the sphere $\rho = 3$.

Problem 172 Let B be a ball of radius R centered at the origin. Calculate $\int\int\int_B e^{-\rho^3}\, dV$

Problem 173 Let $u = \frac{2x}{x^2+y^2}$ and $v = \frac{-2y}{x^2+y^2}$ calculate $\frac{\partial(x,y)}{\partial(u,v)}$.

Problem 174 Suppose $\delta(x,y,z) = 1 = dM/dV$ for $x,y,z > 0$. Find center of mass for a sphere with this density δ centered at $(1,2,3)$.

Problem 175 Suppose $\delta(x,y,z) = xyz = dM/dV$ for $x,y,z > 0$. Find center of mass for a sphere with this density centered at $(1,2,3)$.

Problem 175+i Suppose you have a cylindrical oil tank which is placed on a hill close to your house. After some time the land settles and the oil tank is not level. Suppose you read a dip-stick which is designed for a level-set-up and find the tank is half-full. Suppose the tank is slanted at 20 degrees relative to the true horizontal. In other words, suppose the axis of the cylinder makes and angle of 70 degrees with the vertical. Fortunately, the tank is only tilted along that direction and the perpendicular direction to the central-axis remains at a right angle to the vertical. If you have a 1000gallon tank then how much oil do you really have?
numerical integration is totally fine here, although, this may have a closed-form solution.

Problem 176 Calculate $\int_R \sqrt{x+2y}\sin(x-y)\, dA$ where $R = [0,1]\times[0,1]$ by making an appropriate change of variables.

Problem 177 Find the center of mass for a laminate of variable density $\delta(r,\theta) = r\sin^2(\theta)$ which is bounded by $r = \sin(2\theta)$