Lecture 29: Complex Notation for Electromagnetic Waves

We begin by studying waves of a given angular frequency ω which travel in the z-direction. We also suppose the waves have no x or y dependence... these are <u>monochromatic</u> <u>plane</u> <u>waves</u>

$$\tilde{E}(z,t) = \tilde{E}_0 e^{i(kz-\omega t)}$$

$$\tilde{B}(z,t) = \tilde{B}_0 e^{i(kz-\omega t)}$$

I'm tempted to add a vector symbol since technically both $\tilde{E}_0$ and $\tilde{B}_0$ are vectors in $\mathbb{C}^3$ which record 3 complex amplitudes, I'll try to stick to GRIFFITH's notation...

$$\tilde{E}_0 = (\tilde{E}_0)_x \hat{x} + (\tilde{E}_0)_y \hat{y} + (\tilde{E}_0)_z \hat{z}$$

$$\tilde{B}_0 = (\tilde{B}_0)_x \hat{x} + (\tilde{B}_0)_y \hat{y} + (\tilde{B}_0)_z \hat{z}$$

It seems a mathematical detour is warranted here. What is the calculus of $\mathbb{C}^3$-valued function of space and time?

$$\tilde{F} = \vec{A} + i\vec{B} \quad \text{where } \vec{A}, \vec{B} \text{ are real vector fields}$$

$$\left.\begin{aligned} \frac{\partial \tilde{F}}{\partial t} &= \frac{\partial \vec{A}}{\partial t} + i\frac{\partial \vec{B}}{\partial t} \\ \frac{\partial \tilde{F}}{\partial x_j} &= \frac{\partial \vec{A}}{\partial x_j} + i\frac{\partial \vec{B}}{\partial x_j} \end{aligned}\right\} \text{ partial differentiate component wise Re \& Im separately.}$$

$$\left.\begin{aligned} \nabla \cdot \tilde{F} &= \nabla \cdot \vec{A} + i\nabla \cdot \vec{B} \\ \nabla \times \tilde{F} &= \nabla \times \vec{A} + i\nabla \times \vec{B} \end{aligned}\right\} \text{ complex sol}^{\underline{n}} \text{ of Maxwell's Eq}^{\underline{n}} \text{ will give two real sol}^{\underline{n}}\text{s.}$$

Complex Maxwell's Eq^{n}s (sourceless)

$\nabla \cdot \widetilde{E} = 0$	$\longleftrightarrow$	$\nabla \cdot (\text{Re}\,\widetilde{E}) = 0$ & $\nabla \cdot (\text{Im}\,\widetilde{E}) = 0$
$\nabla \cdot \widetilde{B} = 0$	$\longleftrightarrow$	$\nabla \cdot (\text{Re}\,\widetilde{B}) = 0$ & $\nabla \cdot (\text{Im}\,\widetilde{B}) = 0$
$\nabla \times \widetilde{E} = -\dfrac{\partial \widetilde{B}}{\partial t}$	$\longleftrightarrow$	$\nabla \times (\text{Re}\,\widetilde{E}) = -\dfrac{\partial}{\partial t}(\text{Re}(\widetilde{B}))$ and $\nabla \times (\text{Im}\,\widetilde{E}) = -\dfrac{\partial}{\partial t}(\text{Im}(\widetilde{B}))$
$\nabla \times \widetilde{B} = \mu_0 \epsilon_0 \dfrac{\partial \widetilde{E}}{\partial t}$	$\longleftrightarrow$	$\nabla \times (\text{Re}\,\widetilde{B}) = \mu_0 \epsilon_0 \dfrac{\partial}{\partial t}(\text{Re}(\widetilde{E}))$ $\nabla \times (\text{Im}\,\widetilde{B}) = \mu_0 \epsilon_0 \dfrac{\partial}{\partial t}(\text{Im}(\widetilde{B}))$

From these we derive the complex wave equation,

$$\boxed{\nabla^2 \widetilde{E} = \mu_0 \epsilon_0 \frac{\partial^2 \widetilde{E}}{\partial t^2} \quad\Big|\quad \nabla^2 \widetilde{B} = \mu_0 \epsilon_0 \frac{\partial^2 \widetilde{B}}{\partial t^2}}$$

Observation: the plane waves solve these equations

$$\widetilde{E}(z,t) = \widetilde{E}_0\, e^{i(kz-\omega t)}$$

$$\widetilde{B}(z,t) = \widetilde{B}_0\, e^{i(kz-\omega t)}$$

Consider,

$$\nabla \cdot \widetilde{E} = \widetilde{E}_0\, e^{i(kz-\omega t)}\, \nabla\big(i(kz-\omega t)\big) = ik\, \widetilde{E}_0\, e^{i(kz-\omega t)}\, \hat{z}$$

Consider $\tilde{E} = \tilde{E}_o e^{i(kz-\omega t)}$

$$\nabla \cdot \tilde{E} = \tilde{E}_o \cdot \nabla (e^{i(kz-\omega t)})$$
$$= \langle (\tilde{E}_o)_x, (\tilde{E}_o)_y, (\tilde{E}_o)_z \rangle \cdot \langle 0, 0, ik \rangle e^{i(kz-\omega t)}$$
$$= ike^{i(kz-\omega t)} (\tilde{E}_o)_z$$

Hence $\nabla \cdot \tilde{E} = 0$ forces $(\tilde{E}_o)_z = 0$. We find

$$\boxed{\tilde{E}_o = (\tilde{E}_o)_x \hat{x} + (\tilde{E}_o)_y \hat{y}}$$

Likewise from $\tilde{B} = \tilde{B}_o e^{i(kz-\omega t)}$ we derive from $\nabla \cdot \tilde{B} = 0$ the requirement $(\tilde{B}_o)_z = 0$ thus

$$\boxed{\tilde{B}_o = (\tilde{B}_o)_x \hat{x} + (\tilde{B}_o)_y \hat{y}}$$

NEXT, <u>FARADAY'S LAW</u>, let $\theta = kz - \omega t$

$$\nabla \times \tilde{E} = \nabla \times (\tilde{E}_o e^{i\theta})$$
$$= -\tilde{E}_o \times \nabla e^{i\theta}$$
$$= \nabla e^{i\theta} \times \tilde{E}_o$$
$$= ie^{i\theta} k \hat{z} \times \tilde{E}_o$$

$$\frac{\partial \tilde{B}}{\partial t} = \tilde{B}_o \frac{\partial}{\partial t}\left[e^{i\theta}\right] = -i\omega \tilde{B}_o e^{i\theta}$$

Thus, $ie^{i\theta} k \hat{z} \times \tilde{E}_o = i\omega \tilde{B}_o e^{i\theta} \Rightarrow \boxed{\tilde{B}_o = \frac{k}{\omega} \hat{z} \times \tilde{E}_o}$

It follows that $\boxed{B_o = \frac{k}{\omega} E_o}$ $\tilde{B}_o, \tilde{E}_o$ are in-phase

Maxwell's Correction

$$\nabla \times \tilde{B} = (\nabla e^{i\theta}) \times \tilde{B}_0 \quad \& \quad \frac{\partial \tilde{E}}{\partial t} = -i\omega \tilde{E}_0 e^{i\theta}$$
$$= i e^{i\theta} k \hat{z} \times \tilde{B}_0$$

Consequently, $\nabla \times \tilde{B} = \mu_0 \epsilon_0 \frac{\partial \tilde{E}}{\partial t}$ yields

$$i e^{i\theta} k \hat{z} \times \tilde{B}_0 = -i \mu_0 \epsilon_0 \omega \tilde{E}_0 e^{i\theta}$$

$$\hat{z} \times \tilde{B}_0 = \frac{-\mu_0 \epsilon_0 \omega}{k} \tilde{E}_0$$

$$\tilde{B}_0 \times \hat{z} = \frac{\mu_0 \epsilon_0 \omega}{k} \tilde{E}_0$$

$$\underbrace{\hat{z} \times (\tilde{B}_0 \times \hat{z})} = \frac{\mu_0 \epsilon_0 \omega}{k} \hat{z} \times \tilde{E}_0$$

$$\tilde{B}_0 (\hat{z} \cdot \hat{z}) - \hat{z}(\hat{z} \cdot \tilde{B}_0) = \tilde{B}_0 = \frac{\mu_0 \epsilon_0 \omega}{k} \hat{z} \times \tilde{E}_0 = \frac{k}{\omega} \hat{z} \times \tilde{E}_0$$

Thus $\frac{\mu_0 \epsilon_0 \omega}{k} = \frac{k}{\omega} \Rightarrow \frac{k^2}{\omega^2} = \mu_0 \epsilon_0 = \frac{1}{c^2}$

That is, $\boxed{k = \omega/c}$ or $\boxed{\frac{k}{\omega} = \frac{1}{c}}$ or $\boxed{\frac{\omega}{k} = c}$

$$\boxed{B_0 = \frac{k}{\omega} E_0 = \frac{1}{c} E_0}$$

<u>Remark</u>: Maxwell's Correction $\Rightarrow \frac{k}{\omega} = \sqrt{\mu_0 \epsilon_0}$
or $\frac{\omega}{k} = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$, I believe this connects w dk which sets the <u>speed</u> of the wave at c.

$\boxed{E1}$ Suppose $\tilde{E}(z,t) = \tilde{E}_0 e^{i(kz-\omega t)}\hat{x}$

Ok, apparently I need to add the vector notation

$$\boxed{\vec{\tilde{E}}_0 = \tilde{E}_0 \hat{x}}$$

Notice $\tilde{E}_0$ is complex scalar here.

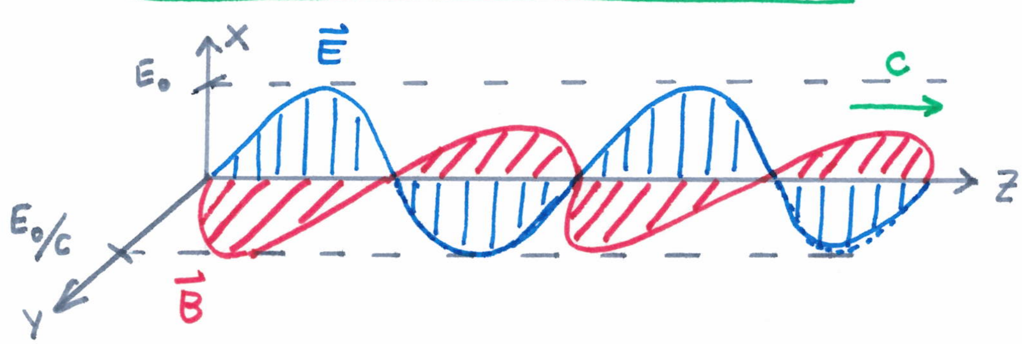
$$\boxed{\vec{\tilde{B}}_0 = \frac{1}{c}\left(\hat{z} \times \vec{\tilde{E}}_0\right) = \frac{\tilde{E}_0}{c}\hat{z}\times\hat{x} = \frac{\tilde{E}_0}{c}\hat{y}}$$

Oh, for $\vec{E}$-field in $\hat{x}$-direction which travels in z-direction we get $\vec{B}$-field waving in $\hat{y}$-direction. That is,

$$\boxed{\begin{aligned} \tilde{E}(z,t) &= \tilde{E}_0 e^{i(kz-\omega t)}\hat{x} \\ \tilde{B}(z,t) &= \left(\tilde{E}_0/c\right) e^{i(kz-\omega t)}\hat{y} \end{aligned}}$$

From which we find physical fields from $Re(\tilde{E})$, $Re(\tilde{B})$

$$\boxed{\begin{aligned} \vec{E}(z,t) &= E_0 \cos(kz-\omega t+\delta)\hat{x} \\ \vec{B}(z,t) &= \frac{E_0}{c}\cos(kz-\omega t+\delta)\hat{y} \end{aligned}}$$



Generally, replacing $k\hat{z}$ with $\vec{k}$ and $\hat{x}$ with $\hat{n}$

$$\boxed{\begin{aligned} \tilde{E}(\vec{r},t) &= \tilde{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\hat{n} \\ \tilde{B}(\vec{r},t) &= \frac{1}{c}\tilde{E}_0(\hat{k}\times\hat{n}) = \frac{1}{c}\hat{k}\times\tilde{E} \end{aligned}}$$

ENERGY AND MOMENTUM IN ELECTROMAGNETIC WAVES

Recall $u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$ gives E&M energy density

Let's calculate this for the monochromatic plane wave

$$\boxed{\begin{aligned} \vec{E}(\vec{r},t) &= E_0 \cos(\vec{k}\cdot\vec{r} - \omega t + \delta)\,\hat{n} \\ \vec{B}(\vec{r},t) &= \frac{1}{c} E_0 \cos(\vec{k}\cdot\vec{r} - \omega t + \delta)\,(\hat{k}\times\hat{n}) \end{aligned}}$$

Observe $B^2 = \frac{1}{c^2} E^2 = \mu_0 \epsilon_0 E^2$ thus,

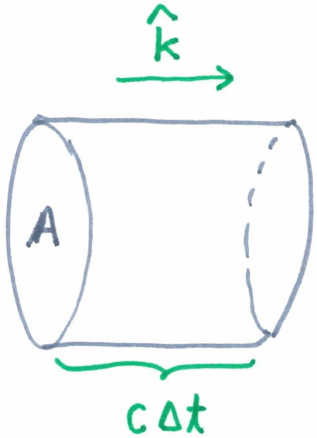
$$u = \frac{1}{2}\left[\epsilon_0 E^2 + \frac{1}{\mu_0}\left[\mu_0 \epsilon_0 E^2\right]\right] \Rightarrow \boxed{u = \epsilon_0 E^2}$$

To be precise, $u(\vec{r},t) = \epsilon_0 E_0^2 \cos^2(\vec{k}\cdot\vec{r} - \omega t + \delta)$. Next the energy flux density (per unit area time)

$$\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = \frac{E_0^2}{\mu_0 c}\cos^2\theta\; \hat{n}\times(\hat{k}\times\hat{n}) \qquad \frac{1}{\mu_0 c} = \frac{c}{\mu_0 c^2} = \frac{c}{\frac{\mu_0}{\mu_0 \epsilon_0}}$$

$$\hat{n}\times(\hat{k}\times\hat{n}) = \hat{k}(\hat{n}\cdot\hat{n}) - \hat{n}(\hat{n}\cdot\hat{k}) = \hat{k} \quad \text{since } \hat{n}\perp\hat{k}$$

$$\boxed{\vec{S} = c\,\epsilon_0 E_0^2 \cos^2(\vec{k}\cdot\vec{r} - \omega t + \delta)\,\hat{k}} \boxed{= c u \hat{k}}$$



$$\Delta U = (A c \Delta t)\, u$$

$$\underbrace{\frac{\Delta U}{A\,\Delta t}}_{\substack{\text{energy} \\ \text{per area/time} \\ \text{flowing}}} = \underbrace{c u}_{\substack{\text{speed of light} \\ \cdot\ \text{energy per volume.}}}$$

Momentum density stored in the fields,

$$\vec{g} = \frac{1}{c^2}\vec{S} = \frac{1}{c^2} c u \hat{k} = \left(\frac{u}{c}\right)\hat{k}$$

$$\boxed{\vec{g}(\vec{r},t) = \frac{\epsilon_0 E_0^2}{c}\cos^2(\vec{k}\cdot\vec{r} - \omega t + \delta)\,\hat{k}}$$

Def^n/ For a periodic function $f(t+T) = f(t)$ with period T the average of f,

$$\langle f \rangle = \frac{1}{T}\int_0^T f(t)\,dt$$

Calculate, for $\theta = \vec{k}\cdot\vec{r} - \omega t + \delta$ where $\omega = \frac{2\pi}{T}$

$$\begin{aligned}
\langle \cos^2\theta \rangle &= \frac{1}{T}\int_0^T \cos^2(\vec{k}\cdot\vec{r} - \omega t + \delta)\,dt \\
&= \frac{1}{T}\int_{\vec{k}\cdot\vec{r}+\delta}^{\vec{k}\cdot\vec{r}+\delta-2\pi} \cos^2\theta \left(\frac{-d\theta}{\omega}\right) \\
&= \frac{1}{2\pi}\int_{\alpha-2\pi}^{\alpha} \frac{1}{2}(1 + \cos(2\theta))\,d\theta \\
&= \frac{1}{4\pi}\left[\alpha - (\alpha - 2\pi)\right] \\
&= \frac{1}{2}.
\end{aligned}$$

Therefore, the time averages for u, $\vec{S}$, $\vec{g}$ are

$\langle u \rangle = \frac{1}{2}\epsilon_0 E_0^2$	$\langle \vec{S} \rangle = \frac{1}{2} c\,\epsilon_0 E_0^2\,\hat{k}$
$\langle \vec{g} \rangle = \frac{1}{2c}\epsilon_0 E_0^2\,\hat{k}$	

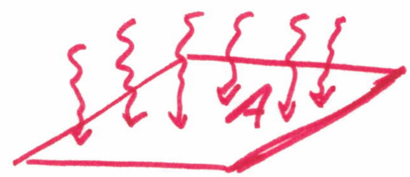
Def^n/ Intensity is the average power per unit area transported by an electromagnetic wave

$$I = \langle S \rangle = \frac{1}{2} c \varepsilon_0 E_0^2$$

To calculate the radiation pressure we need to think through the average force per unit area:

$$P = \frac{F_{em}}{A} = \frac{1}{A}\frac{\Delta P}{\Delta t} = \frac{1}{A}\frac{\langle g \rangle A c \Delta t}{\Delta t} = c\langle g \rangle = \frac{1}{2}\varepsilon_0 E_0^2$$

$$P = \frac{I}{c}$$



P = pressure

Remark: the analysis above assumes perfect absorber with light hitting $\perp$. For perfect reflector the pressure is doubling since ΔP doubles due to the complete turn-around for the light momentum.

ELECTROMAGNETIC WAVES IN MATTER

Inside linear media with permeability μ and permitivity ϵ in the absence of free charge & current, $\vec{D} = \epsilon \vec{E}$ and $\vec{H} = \frac{1}{\mu} \vec{B}$ and Maxwell's Eq$^{\underline{n}}$s read:

$$\boxed{\begin{array}{ll} \nabla \cdot \vec{E} = 0 & \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ \nabla \cdot \vec{B} = 0 & \nabla \times \vec{B} = \mu\epsilon \frac{\partial \vec{E}}{\partial t} \end{array}}$$

We've simply replaced ϵ_0 with ϵ and μ_0 with μ for a homogeneous linear media,

$$\boxed{\begin{array}{l} \text{Def}^{\underline{n}}/ \quad v = \frac{1}{\sqrt{\epsilon\mu}} = \frac{c}{n} \quad \text{where} \quad n = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}} \\ v = \text{speed of light propagation}, \; n = \text{index of refraction} \end{array}}$$

I usually define $n = \frac{c}{v}$ in the elementary E&M course. Griffiths says $\mu \approx \mu_0$ usually so $n \approx \sqrt{\frac{\epsilon}{\epsilon_0}} = \sqrt{\epsilon_r}$ fwiw. The monochromatic plane wave formulas hold with the replacement of ϵ for ϵ_0 and μ for μ_0 and of course v for c,

$$u = \frac{1}{2}\left[\epsilon E^2 + \frac{1}{\mu} B^2\right]$$

$$\vec{S} = \frac{1}{\mu} \vec{E} \times \vec{B}$$

$$\omega = kv$$

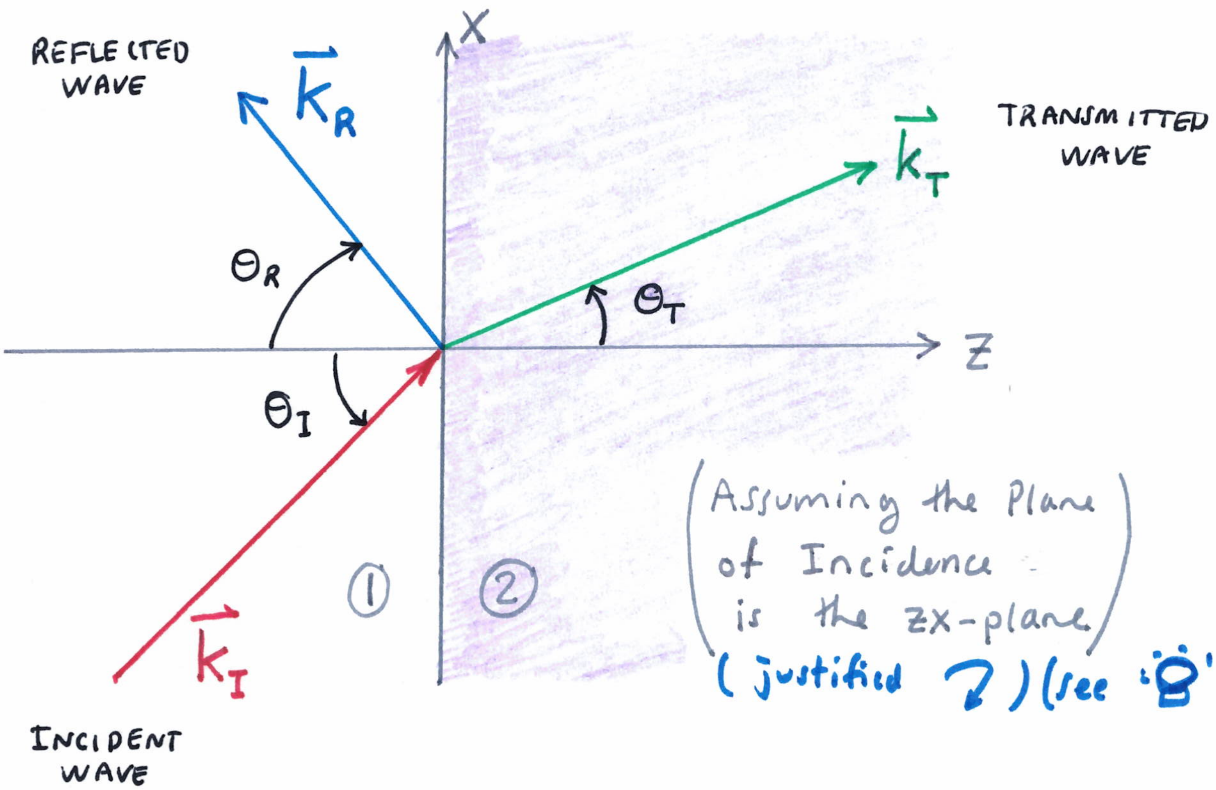
$$B = \frac{1}{v} E$$

$$I = \frac{1}{2} \epsilon v E_0^2$$

Reflection and Transmission of Light

If an electromagnetic wave travels from medium 1 to medium 2 with μ_1, ϵ_1 and μ_2, ϵ_2 respective then we must satisfy boundary conditions:

$$\boxed{\begin{array}{ll} \text{(i.)}\ \ \epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp} & \text{(iii.)}\ \ E_1^{\parallel} = E_2^{\parallel} \\ \text{(ii.)}\ \ B_1^{\perp} = B_2^{\perp} & \text{(iv.)}\ \ \frac{1}{\mu_1} B_1^{\parallel} = \frac{1}{\mu_2} B_2^{\parallel} \end{array}}$$



(Assuming the Plane of Incidence is the zx-plane) (justified ?) (see 🟊)

Monochromatic plane waves in mediums 1 & 2:

$$\begin{cases} \tilde{E}_I(\vec{r},t) = \tilde{E}_{0_I} e^{i(\vec{k}_I \cdot \vec{r} - \omega t)}, & \tilde{B}_I(\vec{r},t) = \frac{1}{v_1}\left(\hat{k}_I \times \tilde{E}_I\right) \\ \tilde{E}_R(\vec{r},t) = \tilde{E}_{0_R} e^{i(\vec{k}_R \cdot \vec{r} - \omega t)}, & \tilde{B}_R(\vec{r},t) = \frac{1}{v_1}\left(\hat{k}_R \times \tilde{E}_R\right) \\ \tilde{E}_T(\vec{r},t) = \tilde{E}_{0_T} e^{i(\vec{k}_T \cdot \vec{r} - \omega t)}, & \tilde{B}_T(\vec{r},t) = \frac{1}{v_2}\left(\hat{k}_T \times \tilde{E}_T\right) \end{cases}$$

Same frequency shared by all 3 waves,

$$k_I v_1 = k_R v_1 = k_T v_2 = \omega, \quad k_I = k_R = \frac{v_2}{v_1} k_T = \frac{n_1}{n_2} k_T$$

Suppose the plane where the media meet is $z=0$ and $z<0$ is media ① with n_1, v_1, ϵ_1 whereas $z>0$ gives ② with n_2, v_2, ϵ_2 etc.

$$\epsilon_1 (E_1)_z = \epsilon_2 (E_2)_z \quad \& \quad (B_1)_z = (B_2)_z$$

However both I and R waves in ① while T in ② hence,

$$\boxed{\epsilon_1 \left[(E_I)_z + (E_R)_z \right] = \epsilon_2 (E_T)_z}_{\text{I}} \quad \& \quad \boxed{(B_I)_z + (B_R)_z = (B_T)_z}_{\text{II}}$$

Then $\parallel$ to plane gives x & y equations,

$$\boxed{\begin{aligned} (E_I)_x + (E_R)_x &= (E_T)_x \\ (E_I)_y + (E_R)_y &= (E_T)_y \end{aligned}}_{\text{III}} \qquad \boxed{\begin{aligned} \frac{1}{\mu_1}\left((B_I)_x + (B_R)_x\right) &= \frac{1}{\mu_2}(B_T)_x \\ \frac{1}{\mu_1}\left[(B_I)_y + (B_R)_y\right) &= \frac{1}{\mu_2}(B_T)_y \end{aligned}}_{\text{IV}}$$

To be clear, the above boundary conditions hold for $\underline{z=0}$. If we impose same conditions for complexified fields then the real part will solve the given BC's as desired.

I.) $$\epsilon_1 \left[\tilde{E}_{0_I} e^{i(\vec{k}_I \cdot \vec{r} - \omega t)} + \tilde{E}_{0_R} e^{i(\vec{k}_R \cdot \vec{r} - \omega t)} \right] \cdot \hat{z} = \epsilon_2 \left[\tilde{E}_{0_T} e^{i(\vec{k}_T \cdot \vec{r} - \omega t)} \right] \cdot \hat{z}$$

$$\Rightarrow \vec{k}_I \cdot \vec{r} = \vec{k}_R \cdot \vec{r} = \vec{k}_T \cdot \vec{r} \quad \text{when } z=0$$

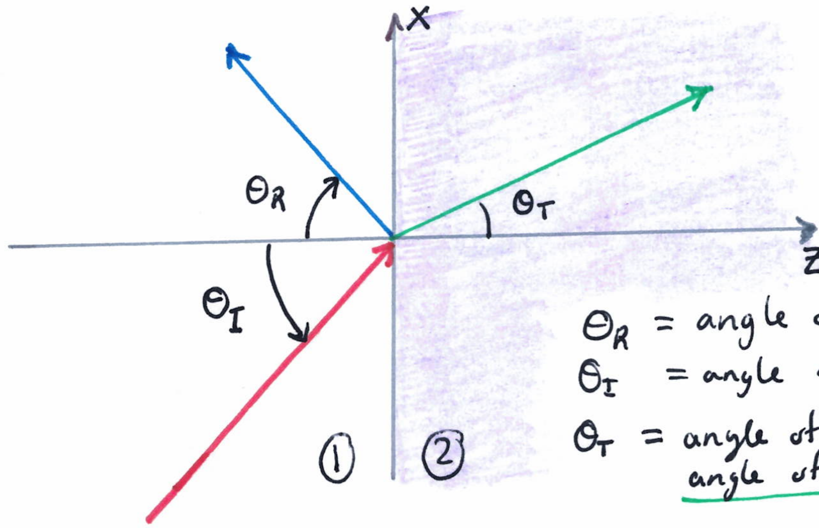
$$\underbrace{x(k_I)_x + y(k_I)_y = x(k_R)_x + y(k_R)_y = x(k_T)_x + y(k_T)_y}_{\text{holds for all } x,y.}$$

$$\Rightarrow \boxed{\begin{aligned} (k_I)_x &= (k_R)_x = (k_T)_x \\ (k_I)_y &= (k_R)_y = (k_T)_y \end{aligned}}$$

we can place k_I in xz-plane then $(\vec{k}_I)_y = 0 \Rightarrow$ both $\vec{k}_R$ and $\vec{k}_T$ are also in xz-plane, thus the plane of incidence exists!

We've shown that:

$\boxed{\text{FIRST LAW: The incident, reflected and transmitted wave vectors form a plane which also includes the normal to the plane (plane of incidence exists)}}$



θ_R = angle of reflection
θ_I = angle of incidence
θ_T = angle of transmission a.k.a angle of refraction

$$(k_I)_x = (k_R)_x = (k_T)_x \Rightarrow k_I \sin\theta_I = k_R \sin\theta_R = k_T \sin\theta_T$$

Recall, $k_I = k_R$ thus $\sin\theta_I = \sin\theta_R \Rightarrow \underline{\theta_I = \theta_R}$.

$\boxed{\text{SECOND LAW: } \theta_I = \theta_R}$

However, $k_I = \frac{n_1}{n_2} k_T$ hence

$$\frac{n_1}{n_2} k_T \sin\theta_I = k_T \sin\theta_T \Rightarrow \underline{n_1 \sin\theta_I = n_2 \sin\theta_T}.$$

$\boxed{\text{THIRD LAW: } \frac{\sin\theta_T}{\sin\theta_I} = \frac{n_1}{n_2} \text{ Snell's Law}}$

Revisiting I, II, III, IV we cancel the exponential terms and find condition on their complex magnitudes,

$$(i.) \quad \epsilon_1 \left(\tilde{E}_{0_I} + \tilde{E}_{0_R} \right)_z = \epsilon_2 \left(\tilde{E}_{0_T} \right)_z$$

$$(ii.) \quad \left(\tilde{B}_{0_I} + \tilde{B}_{0_R} \right)_z = \left(\tilde{B}_{0_T} \right)_z$$

$$(iii.) \quad \left(\tilde{E}_{0_I} + \tilde{E}_{0_R} \right)_{x_i} = \left(\tilde{E}_{0_T} \right)_{x_i} \quad \text{for } i = 1,2 \quad \begin{cases} x_1 = x \\ x_2 = y \end{cases}$$

$$(iv.) \quad \frac{1}{\mu_1} \left(\tilde{B}_{0_I} + \tilde{B}_{0_R} \right)_{x_i} = \frac{1}{\mu_2} \left(\tilde{B}_{0_T} \right)_{x_i} \quad \text{for } i = 1,2$$

where for each wave $\tilde{B}_0 = \frac{1}{v} \hat{k} \times \tilde{E}_0$ and $v = v_1$ or v_2 as appropriate.

Now suppose the polarization lies parallel to the plane, if $\tilde{E}_I$ is in xz-plane then so to is $\tilde{E}_R$, $\tilde{E}_T$. Then (i.) yields:

$$\boxed{\epsilon_1 \left(-\tilde{E}_{0_I} \sin \theta_I + \tilde{E}_{0_R} \sin \theta_R \right) = \epsilon_2 \left(-\tilde{E}_{0_T} \sin \theta_T \right)}$$

Then (iii.) yields, and (iv.),

$$\boxed{\begin{gathered} \tilde{E}_{0_I} \cos \theta_I + \tilde{E}_{0_R} \cos \theta_R = \tilde{E}_{0_T} \cos \theta_T \\ \frac{1}{\mu_1 v_1} \left(\tilde{E}_{0_I} - \tilde{E}_{0_R} \right) = \frac{1}{\mu_2 v_2} \tilde{E}_{0_T} \end{gathered}}$$

ok, $\sin \theta_I = \sin \theta_R$ and so $\tilde{E}_{0_I} - \tilde{E}_{0_R} = \frac{\epsilon_2}{\epsilon_1} \cdot \frac{\sin \theta_T}{\sin \theta_I} \cdot \tilde{E}_{0_T} = \frac{n_1 \epsilon_2}{n_2 \epsilon_1} \tilde{E}_{0_T}$

Hence $\frac{1}{\mu_1 v_1} \cdot \frac{n_1 \epsilon_2}{n_2 \epsilon_1} \cdot \tilde{E}_{0_T} = \frac{1}{\mu_2 v_2} \tilde{E}_{0_T}$, ok, more to point,

$$\left\{ \begin{aligned} &\tilde{E}_{0_I} - \tilde{E}_{0_R} = \beta \tilde{E}_{0_T} \quad \text{where } \beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1} \\ &\tilde{E}_{0_I} + \tilde{E}_{0_R} = \alpha \tilde{E}_{0_T} \quad \text{where } \alpha \equiv \frac{\cos \theta_T}{\cos \theta_I} \end{aligned} \right.$$

algebra

$$\boxed{\tilde{E}_{0_R} = \left(\frac{\alpha - \beta}{\alpha + \beta} \right) \tilde{E}_{0_I}} \quad \text{and} \quad \boxed{\tilde{E}_{0_T} = \left(\frac{2}{\alpha + \beta} \right) \tilde{E}_{0_I}}$$

FRESNEL'S EQUATIONS

Fresnel's Equations for parallel polarization are:

$$\tilde{E}_{0_R} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)\tilde{E}_{0_I} \quad \& \quad \tilde{E}_{0_T} = \left(\frac{2}{\alpha+\beta}\right)\tilde{E}_{0_I}$$

Both $\alpha, \beta > 0$ hence we observe:

(1.) TRANSMITTED AND INCIDENT WAVE ARE IN PHASE.

(2.) REFLECTED WAVE AND INCIDENT ARE

- IN PHASE IF $\alpha > \beta$
- OUT OF PHASE IF $\alpha < \beta$

Also we can solve for α, since $\alpha = \frac{\cos\theta_T}{\cos\theta_I}$,

$$\alpha = \frac{\sqrt{1-\sin^2\theta_T}}{\cos\theta_I} = \frac{\sqrt{1-\left[\frac{n_1}{n_2}\sin\theta_I\right]^2}}{\cos\theta_I}$$

Special Cases

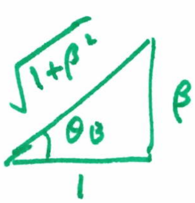
(1.) $\underbrace{\theta_I = 0, \ \alpha = 1}_{\text{normal incidence}}$

(2.) $\theta_I = 90°$, α diverges, total reflections.

(3.) BREWSTER'S ANGLE when $\alpha = \beta$, $\beta = \frac{\mu_1 n_2}{\mu_2 n_1} \approx \frac{n_2}{n_1}$ typically

$$\sin^2\theta_B = \frac{1-\beta^2}{(n_1/n_2)^2 - \beta^2}$$

$$= \frac{1-\beta^2}{\frac{1}{\beta^2} - \beta^2} = \frac{\beta^2(1/\beta^2 - 1)}{\left(\frac{1}{\beta^2}-1\right)(1+\beta^2)} = \frac{\beta^2}{1+\beta^2}$$



$$\frac{\sin\theta_B}{\cos\theta_B} \simeq \frac{\beta}{1} = \frac{n_2}{n_1} \quad \therefore \quad \boxed{\tan\theta_B \approx \frac{n_2}{n_1}}$$

Finally, we should analyze power transmission,

Power per unit area striking the interface is $\vec{S} \cdot \hat{z}$

intensities
$$\begin{cases} I_I = \frac{1}{2} \epsilon_1 v_1 E_{0_I}^2 \cos\theta_I \\ I_R = \frac{1}{2} \epsilon_1 v_1 E_{0_R}^2 \cos\theta_R \\ I_T = \frac{1}{2} \epsilon_2 v_2 E_{0_T}^2 \cos\theta_T \end{cases}$$

Consequently,

$$R \equiv \frac{I_R}{I_I} = \left(\frac{E_{0_R}}{E_{0_I}}\right)^2 = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2$$

$$T \equiv \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left(\frac{E_{0_T}}{E_{0_I}}\right)^2 \frac{\cos\theta_T}{\cos\theta_I} = \alpha\beta\left(\frac{2}{\alpha+\beta}\right)^2$$

You can check, $R + T = 1$. Also, for θ_B where $\alpha = \beta$ we have $R = 0$ and $T = 1$.

Remark: PROBLEM 9.17 on $\perp$ polarization is pretty interesting.