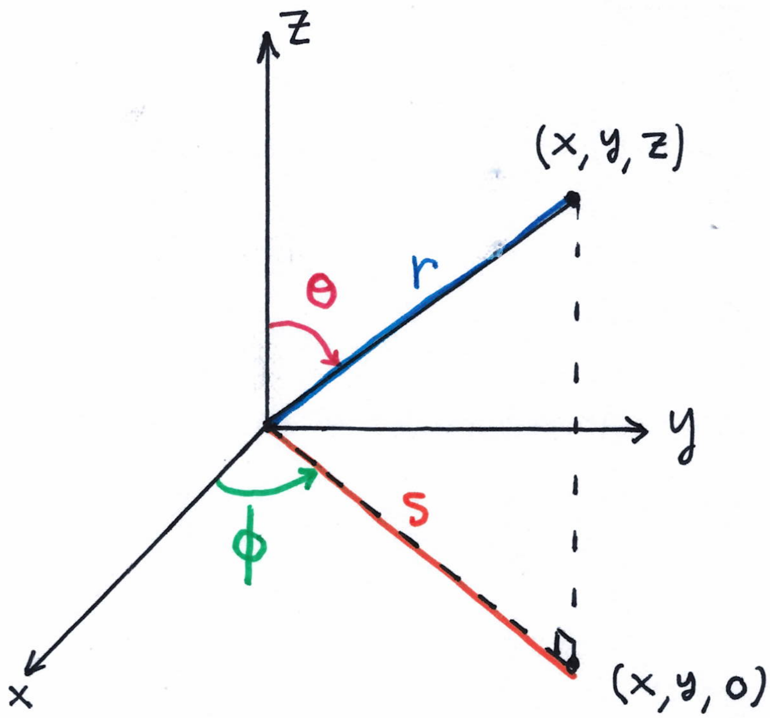
Lecture 2 : Coordinate Frames & Calculus

Most courses in Calculus III fail to explain how grad, curl and divergence are calculated in non-Cartesian coordinates. We follow Griffiths and supply the missing formulas.



Physics Convention

$0 \leq \theta \leq \pi$

$0 \leq \phi \leq 2\pi$

$r = \sqrt{x^2 + y^2 + z^2}$

$s = \sqrt{x^2 + y^2}$

$s^2 + z^2 = r^2$

$s = r \sin\theta$

For $s \neq 0$, $\tan\phi = \frac{y}{x}$

$$x = r\cos\phi \sin\theta = s\cos\phi$$

$$y = r\sin\phi \sin\theta = s\sin\phi$$

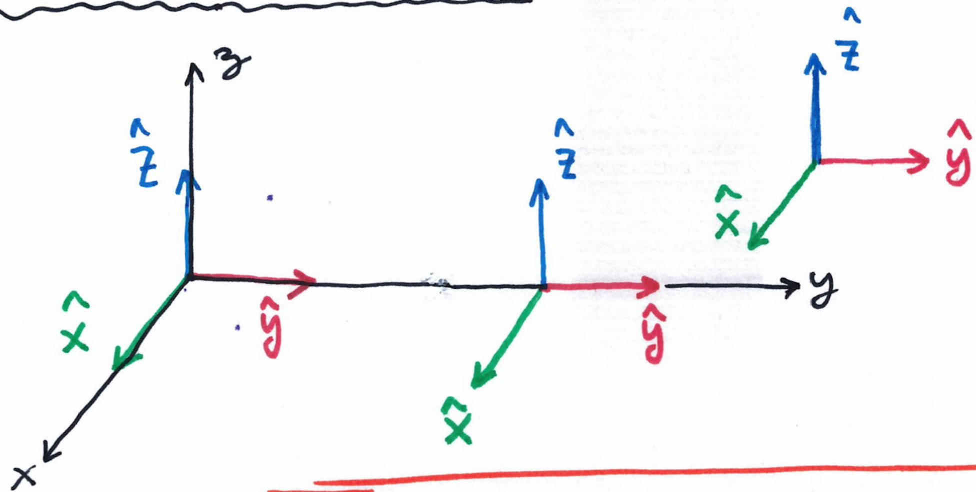
$$z = r\cos\theta$$

Spherical coordinates : r, θ, ϕ

Cylindrical coordinates : s, ϕ, z

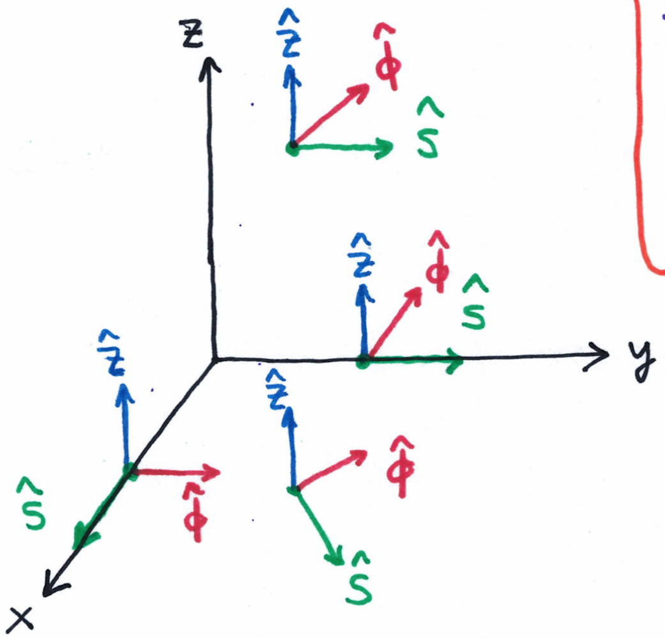
Coordinate Frames:

Cartesian: $\hat{x} \times \hat{y} = \hat{z}$



Remark: $\boxed{\hat{x} = \langle 1,0,0\rangle, \quad \hat{y} = \langle 0,1,0\rangle, \quad \hat{z} = \langle 0,0,1\rangle}$ are constant over all $\mathbb{R}^3$ so considering the point of attachment is not too subtle... but, this parallel transport of the Cartesian frame has to be taught.

Cylindrical: $\hat{s} \times \hat{\phi} = \hat{z}$



$$\boxed{\begin{aligned} \hat{z} &= \langle 0,0,1\rangle \\ \hat{s} &= \langle \cos\phi, \sin\phi, 0\rangle \\ \hat{\phi} &= \langle -\sin\phi, \cos\phi, 0\rangle \end{aligned}}$$

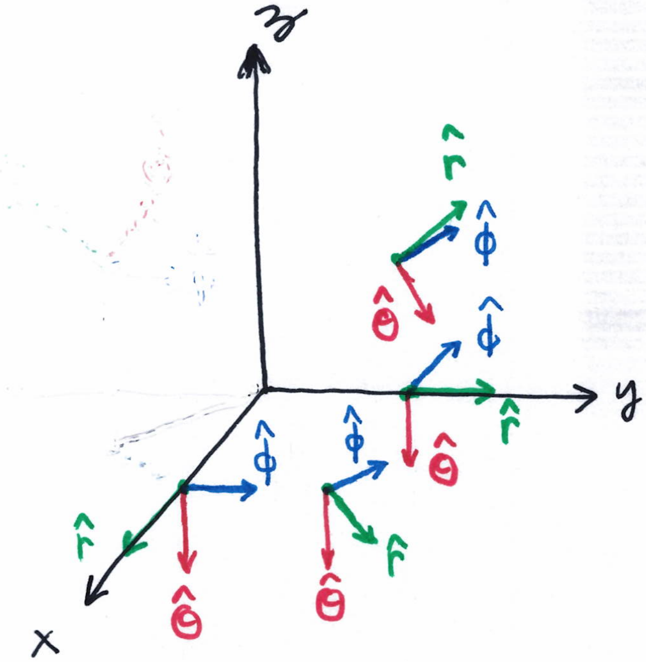
(still easy)

These are orthonormal and you can verify

$$\hat{s} \times \hat{\phi} = \hat{z}$$

Now the point of attachment matters. Notice $\hat{s}$ not defined on z-axis.

SPHERICAL : $\hat{r} \times \hat{\theta} = \hat{\phi}$



$$\boxed{\hat{\phi} = \langle -\sin\phi, \cos\phi, 0 \rangle}$$

same as in Cylindrical context.

$$\boxed{\hat{r} = \langle \cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta \rangle}$$

can see from $r = 1$ in spherical coord. formulas

$$x = r\cos\phi \sin\theta$$
$$y = r\sin\phi \sin\theta$$
$$z = r\cos\theta$$

It is geometrically evident that we have that $\hat{\phi} \times \hat{r} = \hat{\theta}$. Calculate,

$$\hat{\phi} \times \hat{r} = \langle -\sin\phi, \cos\phi, 0 \rangle \times \langle \cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta \rangle$$

$$= \langle \cos\phi \cos\theta, \sin\phi \cos\theta, -\sin^2\phi \sin\theta - \cos^2\phi \sin\theta \rangle$$

$$= \langle \cos\phi \cos\theta, \sin\phi \cos\theta, -\sin\theta \rangle$$

$$\therefore \boxed{\hat{\theta} = \langle \cos\phi \cos\theta, \sin\phi \cos\theta, -\sin\theta \rangle}$$

We should verify $\hat{r} \cdot \hat{r} = \hat{\phi} \cdot \hat{\phi} = \hat{\theta} \cdot \hat{\theta} = 1$ thus $\{\hat{r}, \hat{\theta}, \hat{\phi}\}$ forms a right-handed frame.

Remark: p. 177-178 have computer generated pictures of the cylindrical & spherical frame in math notation r, θ, z or ρ, ϕ, θ in my Fall 25 calc III notes.

FRAMES VIA CALCULUS:

I've shown how to see $\underbrace{\hat{x}, \hat{y}, \hat{z}}_{\text{CART. FRAME}}$ and $\underbrace{\hat{s}, \hat{\phi}, \hat{z}}_{\text{CYLINDRICAL FRAME}}$ and $\underbrace{\hat{r}, \hat{\theta}, \hat{\phi}}_{\text{SPHERICAL FRAME}}$ geometrically. There is another path: CALCULUS

> Thm/ If $u: \mathbb{R}^3 \to \mathbb{R}$ is a function (coordinate function) then ∇u points in direction of increasing u and hence $\hat{u} = \frac{1}{\|\nabla u\|}\nabla u$

CARTESIANS

$$\nabla x = \left\langle \frac{\partial x}{\partial x}, \frac{\partial x}{\partial y}, \frac{\partial x}{\partial z} \right\rangle = \langle 1, 0, 0 \rangle = \hat{x}$$
$$\nabla y = \langle 0, 1, 0 \rangle = \hat{y}$$
$$\nabla z = \langle 0, 0, 1 \rangle = \hat{z}$$

it just so happens $\|\nabla x\| = \|\nabla y\| = \|\nabla z\|$ so division by $\|\nabla u\|$ is not interesting.

CYLINDRICAL

We take an indirect approach to reduce suffering,

$$s^2 = x^2 + y^2$$

chain rule!
$$\begin{cases} \nabla(s^2) = \nabla(x^2+y^2) \\ 2s\,\nabla s = \langle 2x, 2y, 0\rangle \quad \therefore \quad \nabla s = \dfrac{\langle 2x, 2y, 0\rangle}{2s} \end{cases}$$

$$\Rightarrow \nabla s = \left\langle \frac{x}{s}, \frac{y}{s}, 0 \right\rangle$$

$$\Rightarrow \nabla s = \boxed{\underbrace{\langle \cos\phi, \sin\phi, 0\rangle}_{\text{unit-vector!}} = \hat{s}}$$

CYLINDRICAL CONTINUED

$$\boxed{\nabla z = \langle 0, 0, 1\rangle = \hat{z}}$$ same as Cartesians.

Next to find $\hat{\phi}$ we need formula to differentiate

$$\tan\phi = \frac{y}{x}$$

$$\nabla(\tan\phi) = \nabla\left(\frac{y}{x}\right)$$

$$\sec^2\phi\,\nabla\phi = \left\langle \frac{\partial}{\partial x}\left[\frac{y}{x}\right], \frac{\partial}{\partial y}\left[\frac{y}{x}\right], \frac{\partial}{\partial z}\left[\frac{y}{x}\right]\right\rangle$$

$$\nabla\phi = \cos^2\phi\left\langle \frac{-y}{x^2}, \frac{1}{x}, 0\right\rangle$$

$$\nabla\phi = \frac{\cos^2\phi}{x^2}\langle -y, x, 0\rangle$$

but $x = s\cos\phi$

hence $\dfrac{\cos\phi}{x} = \dfrac{1}{s}$

$$\nabla\phi = \frac{1}{s^2}\langle -y, x, 0\rangle$$

$$\|\nabla\phi\| = \frac{1}{s^2}\sqrt{(-y)^2 + x^2} = \frac{1}{s^2}\cdot s = \frac{1}{s}$$

$$\therefore\ \hat{\phi} = \frac{\nabla\phi}{\|\nabla\phi\|} = \frac{\frac{1}{s^2}\langle -y, x, 0\rangle}{\frac{1}{s}} = \left\langle \frac{-y}{s}, \frac{x}{s}, 0\right\rangle$$

$$\therefore\ \boxed{\hat{\phi} = \langle -\sin\phi, \cos\phi, 0\rangle}$$ yep.

SPHERICALS:

$$r^2 = x^2 + y^2 + z^2$$

$$\underbrace{\nabla(r^2)}_{\text{chain rule}} = \nabla(x^2 + y^2 + z^2) = \langle 2x, 2y, 2z\rangle$$

$$2r\,\nabla r = \langle 2x, 2y, 2z\rangle$$

unit-vector

$$\boxed{\nabla r = \left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right\rangle = \underbrace{\langle \cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta\rangle}_{\hat{r}}}$$

SPHERICALS CONTINUED

We could derive $\boxed{\hat{\phi} = \langle -\sin\phi, \cos\phi, 0\rangle}$, but we already have. So let us study $\hat{\theta}$, we need a formula to differentiate, I like the look of $z = r\cos\theta$, it's simple,

$$\nabla z = \nabla(r\cos\theta) \quad \text{product rule!}$$

$$\langle 0,0,1\rangle = (\nabla r)\cos\theta + r(\nabla\cos\theta) \quad \text{chain-rule}$$

$$\langle 0,0,1\rangle = (\cos\theta)\nabla r - r\sin\theta\,\nabla\theta$$

$$\therefore \quad \nabla\theta = \frac{\cos\theta\,\nabla r - \langle 0,0,1\rangle}{r\sin\theta}$$

$$\nabla\theta = \frac{1}{r\sin\theta}\Big\langle \cos\theta\cos\phi\sin\theta,\ \cos\theta\sin\phi\sin\theta,\ \underbrace{\cos^2\theta - 1}_{-\sin^2\theta}\Big\rangle$$

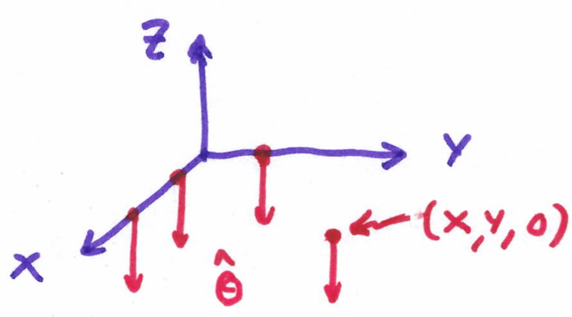
$$\nabla\theta = \frac{1}{r}\underbrace{\langle \cos\theta\cos\phi,\ \sin\phi\cos\theta, -\sin\theta\rangle}_{\text{this is a unit-vector}} \qquad \text{nice cancellation.}$$

$$\therefore \quad \boxed{\hat{\theta} = \langle \cos\theta\cos\phi,\ \sin\phi\cos\theta, -\sin\theta\rangle}$$

CHECK FORMULA

xy-plane has $z = 0$ and $\theta = \pi/2$

thus $\hat{\theta} = \langle 0, 0, -\sin\frac{\pi}{2}\rangle = -\hat{z}$



In the back cover of Griffiths we're given how to express $\hat{x}, \hat{y}, \hat{z}$ in terms of $\hat{r}, \hat{\theta}, \hat{\phi}$ or $\hat{s}, \hat{\phi}, \hat{z}$. We've derived half of those formulas. Let me show how to derive the other half,

$$\boxed{\textbf{CLAIM: } \hat{x} = \sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}}$$

$$\hat{x} = (\hat{x}\cdot\hat{r})\hat{r} + (\hat{x}\cdot\hat{\theta})\hat{\theta} + (\hat{x}\cdot\hat{\phi})\hat{\phi}$$
$$= (\cos\phi\sin\theta)\hat{r} + (\cos\theta\cos\phi)\hat{\theta} - (\sin\phi)\hat{\phi}$$

look back at 5 and 6 to calculate the dot-products.

$$\boxed{\begin{aligned} &\textbf{CLAIM: } \hat{y} = \sin\theta\sin\phi\,\hat{r} + \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi} \\ &\hat{z} = (\cos\theta)\hat{r} - (\sin\theta)\hat{\theta} \\ &\hat{x} = (\cos\phi)\hat{s} - (\sin\phi)\hat{\phi} \\ &\hat{y} = (\sin\phi)\hat{s} + (\cos\phi)\hat{\phi} \end{aligned}}$$

proved much in same way as the $\hat{x}$-calculation

Remark: we can use these frames to calculate line and surface integrals that are natural to these coordinates.

- If a surface is given by fixing a coord then the remaining pair serve as parameters
- If a curve is given by fixing two coord then the curve is parametrized by the remaining coord.

sphere : $r = R_0$; θ, ϕ parameters

cone : $\theta = \theta_0$; r, ϕ parameters

half-plane : $\phi = \phi_0$; r, θ parameters (OR s, z param.)

cylinder : $s = s_0$; z, ϕ parameters

horizontal plane : $z = z_0$; x, y parameters

plane : $y = y_0$; z, x parameters

plane : $x = x_0$; y, z parameters.

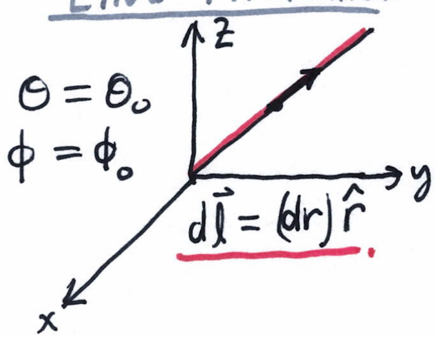
To calculate $\int_C \vec{F} \cdot d\vec{l}$ or $\int_S \vec{F} \cdot d\vec{a}$ we either need a parametrization of C or S, or we need C or S to fit nicely with our favorite coordinant systems. Both $d\vec{l}$ and $d\vec{a}$ can be built as described on p. 38–39 of Griffiths.

SPHERICAL :

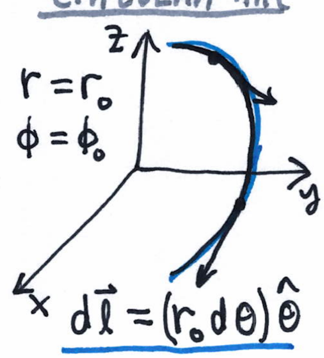
$$\left.\begin{aligned} dl_r &= dr \\ dl_\theta &= r\,d\theta \\ dl_\phi &= r\sin\theta\,d\phi \end{aligned}\right\} \quad d\vec{l} = (dl_r)\hat{r} + (dl_\theta)\hat{\theta} + (dl_\phi)\hat{\phi}$$

$$\boxed{d\vec{l} = (dr)\hat{r} + (r\,d\theta)\hat{\theta} + (r\sin\theta\,d\phi)\hat{\phi}}$$

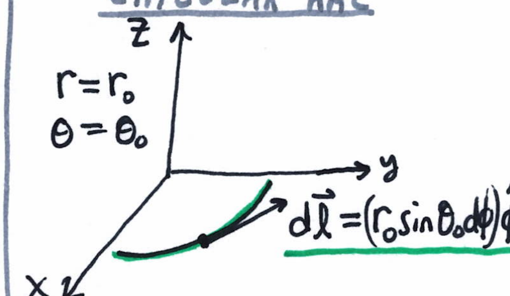
LINE FROM ORIGIN

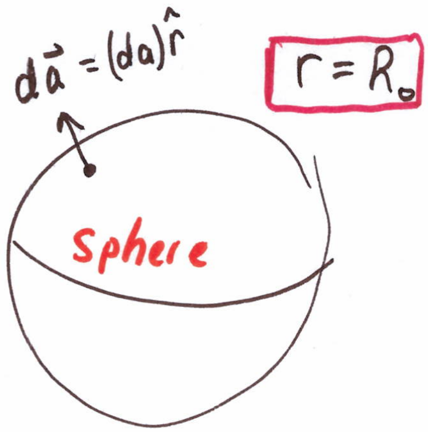


CIRCULAR ARC



CIRCULAR ARC

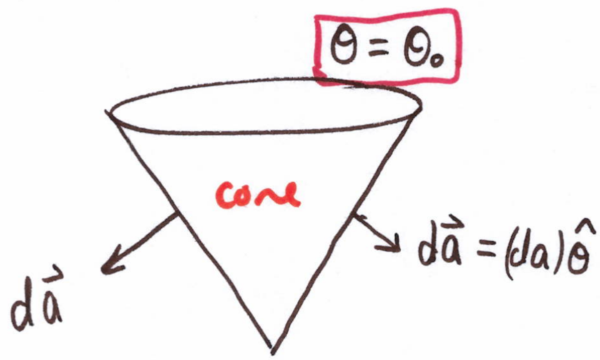




$$d\vec{a} = (dl_\theta \, dl_\phi)\,\hat{r}$$

$$\boxed{d\vec{a} = (r^2 \sin\theta \, d\theta \, d\phi)\,\hat{r} \quad \text{where } r = R_o}$$

area element for sphere



$$d\vec{a} = (dl_\phi \, dl_r)\,\hat{\theta}$$

$$\boxed{d\vec{a} = (r \sin\theta \, d\phi \, dr)\,\hat{\theta}}$$

area element for cone

CYLINDRICALS

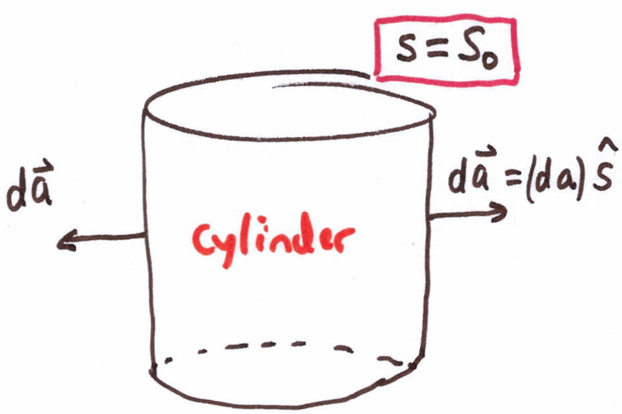
$dl_s = ds$

$dl_\phi = s\,d\phi$

$dl_z = dz$

$$d\vec{l} = (dl_s)\,\hat{s} + (dl_\phi)\,\hat{\phi} + (dl_z)\,\hat{z}$$

$$\boxed{d\vec{l} = (ds)\,\hat{s} + (s\,d\phi)\,\hat{\phi} + (dz)\,\hat{z}}$$



$$d\vec{a} = (dl_\phi \, dl_z)\,\hat{s}$$

$$\boxed{d\vec{a} = (s\,d\phi\,dz)\,\hat{s}}$$