LECTURE 2: SETS AND THEIR PROPERTIES:

- A set with <u>no</u> elements is the <u>empty set</u>: $\{\ \} = \emptyset$ ← notation for empty set
- A set with <u>one</u> element is a <u>singleton</u>: e.g. $\{0\}$

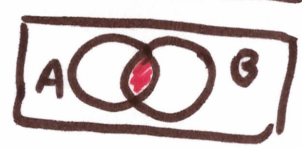
Defⁿ/ If a set S has n-elements where $n \in \mathbb{N}$ then $|S| = n$ and S is <u>finite</u>. Also, $|\emptyset| = 0$ and $\emptyset$ is finite. If S is not finite then S is infinite and $|S| = \infty$.

- I usually write $A \subseteq B$, but Manetti uses $A \subset B$ to mean $A \subseteq B$. When $A \neq B$ and $A \subset B$ then Manetti writes $A \subsetneq B$. Forgive me if I ignore the text and stick with my usual practice ☺.

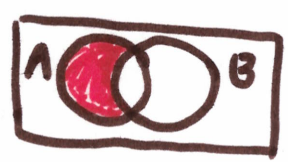
Defⁿ/ If A, B are sets and $x \in A \Rightarrow x \in B$ for each $x \in A$ then we write $A \subseteq B$ and say A is <u>subset</u> of B whereas B is <u>superset</u> of A

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$ <u>union</u>

$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$ <u>intersection</u>



$A - B = \{x \in A \mid x \notin B\}$ <u>set difference</u>



$\boxed{E1}$ $\mathbb{N} = \{1, 2, 3, \ldots\}$: natural #'s

$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$: integers

$\mathbb{Q} = \{ \frac{p}{q} \mid p, q \in \mathbb{Z}, \ q \neq 0 \}$: rational #'s

$\mathbb{R} = (-\infty, \infty)$: real #'s

$\mathbb{C} = \{ a + ib \mid a, b \in \mathbb{R}, \ i^2 = -1 \}$: complex #'s

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

in fact,

$$\underbrace{\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}}_{\text{these are all strict inclusions.}}$$

Def^{n}/ For sets X, Y a <u>function</u> or <u>map</u> $f : X \longrightarrow Y$ is a single-valued assignment of $f(x)$ for each $x \in X$. We also write $x \longmapsto f(x)$ or $x \longmapsto y$ to express the action of f and $X \xrightarrow{f} Y$. $X = \text{domain}(f)$ and $Y = \text{codomain}(f)$. Let $A \subseteq X$ and $B \subseteq Y$ then

$$f(A) = \{ f(x) \mid x \in A \} = \{ y \in B \mid \exists x \in A \text{ such that } y = f(x) \}$$

$$f^{-1}(B) = \{ x \in A \mid f(x) \in B \}$$

image of A under f

inverse image of B under f

$\boxed{E2}$ $f(x, y) = x^2 + y^2$ defines $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$f^{-1}((-\infty, 0)) = \emptyset$ since $x^2 + y^2 \geq 0$

$f^{-1}\{0\} = \{(x, y) \mid x^2 + y^2 = 0\} = \{(0, 0)\}$

$f^{-1}\{R^2\}$ = CIRCLE OF RADIUS R for $R > 0$.

PROPERTIES OF SETS

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
distributive laws

$$A - (B \cup C) = (A - B) \cap (A - C)$$
$$A - (B \cap C) = (A - B) \cup (A - C)$$
De Morgan's laws

Def^{n}/ If U is the universal set then the complement of $X \subseteq U$ is defined by $\overline{X} = U - X$

$$\overline{B \cup C} = \overline{B} \cap \overline{C}$$
$$\overline{B \cap C} = \overline{B} \cup \overline{C}$$

$A \cup B = B \cup A$
$A \cup (B \cup C) = (A \cup B) \cup C$
$A \subseteq A \cup B$
$A \cup A = A$
$A \cup \phi = A$
$A \subseteq B \Longleftrightarrow A \cup B = B$

Properties of union

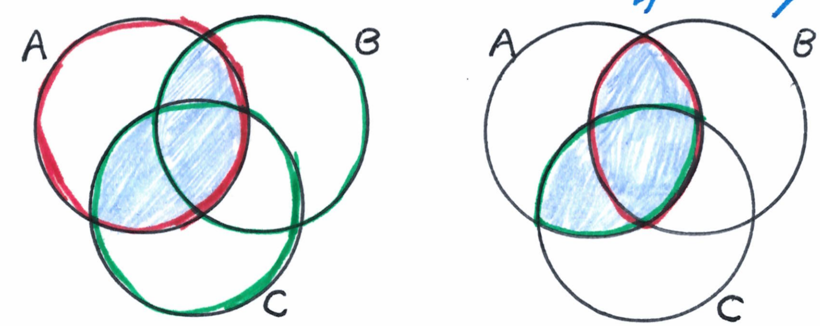
$A \cap B = B \cap A$
$A \cap (B \cap C) = (A \cap B) \cap C$
$A \cap B \subseteq A$
$A \cap A = A$
$A \cap \phi = \phi$
$A \subseteq B \Longleftrightarrow A \cap B = A$

Properties of intersection

$\boxed{A = B \Longleftrightarrow A \subseteq B \text{ and } B \subseteq A}$ ← double containment $\Longleftrightarrow$ set equality

$\boxed{A \subseteq B \text{ and } B \subseteq C \Rightarrow A \subseteq C}$

Proof by Picture



$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

CLAIM: $A-(B\cup C)=(A-B)\cap(A-C)$

Proof: Let $x\in A-(B\cup C)$ then $x\in A$ and $x\notin B\cup C$. Thus $x\notin B$ and $x\notin C$ (if $x\in B$ or $x\in C$ then $x\in B\cup C$). Consequently, $x\in A$ and $x\notin B$ and $x\notin C$ and we find $x\in A-B$ and $x\in A-C$. Thus $x\in(A-B)\cap(A-C)$. This shows $A-(B\cup C)\subseteq(A-B)\cap(A-C)$.

Conversely, suppose $x\in(A-B)\cap(A-C)$ then $x\in A-B$ and $x\in A-C$. Therefore, $x\in A$ and $x\notin B$ and $x\notin C$. It follows $x\notin B\cup C$ hence $x\in A-(B\cup C)$ and we find $(A-B)\cap(A-C)\subseteq A-(B\cup C)$. In conclusion, $A-(B\cup C)=(A-B)\cap(A-C)$. //

CLAIM: If $A\subseteq B$ and $B\subseteq C$ then $A\subseteq C$

Proof: suppose $A\subseteq B$ and $B\subseteq C$. Let $x\in A$ then as $A\subseteq B$ we find $x\in B$. But, as $B\subseteq C$ and $x\in B$ we find $x\in C$ $\therefore$ $A\subseteq C$. //

Def^n/ Let $\mathcal{J}$ be a set of <u>indices</u> then let A_i be a set for each $i \in \mathcal{J}$. We define:

$$\bigcup_{i \in \mathcal{J}} A_i = \{ x \mid \exists j \in \mathcal{J} \text{ such that } x \in A_j \}$$

$$\bigcap_{i \in \mathcal{J}} A_i = \{ x \mid x \in A_j \text{ for all } j \in \mathcal{J} \}$$

When $\mathcal{J} = \mathbb{N}$ we write $\bigcup_{i \in \mathcal{J}} A_i = \bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i \in \mathcal{J}} A_i = \bigcap_{i=1}^{\infty} A_i$

CLAIM: $\left(\bigcap_{i \in I} A_i \right) \cup \left(\bigcap_{j \in J} B_j \right) = \bigcap_{\substack{i \in I \\ j \in J}} (A_i \cup B_j)$

Proof/2: Let $x \in \left(\bigcap_I A_i \right) \cup \left(\bigcap_J B_j \right)$ then $x \in \bigcap_I A_i$ or $x \in \bigcap_J B_j$.

① If $x \in \bigcap_I A_i$ then $x \in A_i$ for all $i \in I$ thus $x \in A_i \cup B_j$ $\forall i \in I, j \in J$ thus $x \in \bigcap_{I,J} (A_i \cup B_j)$.

② If $x \in \bigcap_J B_j$ then $x \in B_j$ for all $j \in J$ thus $x \in A_i \cup B_j$ $\forall i \in I, j \in J$ thus $x \in \bigcap_{I,J} (A_i \cup B_j)$.

Therefore, $\left(\bigcap_I A_i \right) \cup \left(\bigcap_J B_j \right) \subseteq \bigcap_{I,J} (A_i \cup B_j)$ // (you can reverse the argument to prove the other inclusion)

PROPERTIES OF IMAGES & INVERSE IMAGES

$$f(A \cup B) = f(A) \cup f(B)$$
$$f(A \cap B) \subseteq f(A) \cap f(B)$$
$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$$
$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$$
$$f(A \cap f^{-1}(B)) = f(A) \cap B \quad \text{(Projection formula)}$$

- Can replace the unions and intersections over A and B to same relation over families of indexed sets

Proof: Let $x \in f(A \cup B)$ then $\exists y \in A \cup B$ for which $f(y) = x$.

Note $y \in A \cup B$ means $y \in A$ or $y \in B$ thus $x \in f(A)$ or $x \in f(B)$ and we conclude $x \in f(A) \cup f(B)$ $\therefore$ $f(A \cup B) \subseteq f(A) \cup f(B)$.

Let $z \in f(A) \cup f(B)$ then $z \in f(A)$ or $z \in f(B)$. If $z \in f(A)$ then $\exists w \in A$ s.t. $f(w) = z$ thus $w \in A \cup B$ and $f(w) = z$ hence $z \in f(A \cup B)$. Likewise $z \in f(B) \Rightarrow z \in f(A \cup B)$ and we conclude $f(A) \cup f(B) \subseteq f(A \cup B)$.

Therefore, $f(A \cup B) = f(A) \cup f(B)$.

Claim: $f(A \cap f^{-1}(B)) \subseteq f(A) \cap B$

Proof: Let $x \in f(A \cap f^{-1}(B))$ then $\exists y \in A \cap f^{-1}(B)$ for which $f(y) = x$. Now $y \in A \cap f^{-1}(B)$ means $y \in A$ and $y \in f^{-1}(B)$ thus $\exists b \in B$ such that $f(y) = b$. It follows $f(y) = b = x \in B$ and $x \in f(A)$ consequently $x \in f(A) \cap B$. //

CAUTION: f^{-1} is not generally a function thus $f^{-1}(\text{point}) = ?$ whereas $f^{-1}(\text{set})$ = set of pts in domain which map to the set

• Exploration: if $A = \text{dom}(f)$ then $f^{-1}(B) \cap \text{dom}(f) = f^{-1}(B)$ and the projection formula reduces to $f(f^{-1}(B)) = f(\text{dom}(f)) \cap B$. Then if $B \subseteq f(\text{dom}(f))$ we get $f(f^{-1}(B)) = B$.

Def^n/ $f : X \to Y$ is injective or 1-1 then write $f : X \hookrightarrow Y$ or $X \overset{f}{\hookrightarrow} Y$
$f : X \to Y$ is surjective or onto then write $f : X \twoheadrightarrow Y$ or $X \overset{f}{\twoheadrightarrow} Y$

• If $f : X \to Y$ is injective then $f^{-1} : f(X) \to X$ is a function.

Def^{n}/ <u>CARTESIAN PRODUCT</u> of $\mathcal{X}_1, \dots, \mathcal{X}_n$ sets is

$$\prod_{i=1}^{n} \mathcal{X}_i = \mathcal{X}_1 \times \cdots \times \mathcal{X}_n = \{ (x_1, \dots, x_n) \mid x_i \in \mathcal{X}_i \;\; \forall i = 1, 2, \dots, n \}$$

it is the set of ordered n-tuples. Key property of n-tuples

$$(x_1, \dots, x_n) = (y_1, \dots, y_n) \iff x_1 = y_1, \dots, x_n = y_n$$

<u>CLAIM</u>: $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$

<u>Proof</u>$_2$: Let $x \in (A \cap B) \times (C \cap D)$ then $\exists\, x_1 \in A \cap B$ and $x_2 \in C \cap D$ for which $x = (x_1, x_2)$. But, $x_1 \in A \cap B$ gives $x_1 \in A$ <u>and</u> $x_1 \in B$. Likewise, $x_2 \in C \cap D$ gives $x_2 \in C$ <u>and</u> $x_2 \in D$. Hence

$x = (x_1, x_2) \in A \times C$ and $x = (x_1, x_2) \in B \times D$ $\therefore$ $x \in (A \times C) \cap (B \times D)$.

(I leave the other half of the proof to the reader)

Def^{n}/ Let $A \subseteq \mathcal{X} \times \mathcal{Y}$ and $\emptyset \neq B \subseteq \mathcal{Y}$ then the <u>quotient</u> $(A : B) \subseteq \mathcal{X}$ is defined as $(A : B) = \{ x \in \mathcal{X} \mid \{x\} \times B \subseteq A \}$

$\boxed{E3}$ $(\mathcal{X} \times \mathcal{Y} : \mathcal{Y}) = \mathcal{X}$