Lecture 30: Electromagnetic Waves with Constraints

To begin we study electromagnetic waves inside a conductor. In this context the current density $\vec{J}_f$ need not be zero. Rather,

$$\vec{J}_f = \sigma \vec{E}$$

We face Maxwell's Equations,

$$\text{(i.)} \quad \nabla \cdot \vec{E} = \frac{1}{\epsilon}\rho_f \qquad \text{(iii.)} \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\text{(ii.)} \quad \nabla \cdot \vec{B} = 0 \qquad \text{(iv.)} \quad \nabla \times \vec{B} = \mu\sigma \vec{E} + \mu\epsilon \frac{\partial \vec{E}}{\partial t}$$

Continuity equation for free charge,

$$\nabla \cdot \vec{J}_f = -\frac{\partial \rho_f}{\partial t}.$$

Consequently,

$$\nabla \cdot \vec{E} = \frac{\rho_f}{\epsilon} \Rightarrow \nabla \cdot \left(\frac{\vec{J}_f}{\sigma}\right) = \frac{\rho_f}{\epsilon}$$

$$\Rightarrow \frac{-1}{\sigma}\frac{\partial \rho_f}{\partial t} = \frac{1}{\epsilon}\rho_f$$

$$\Rightarrow \boxed{\frac{\partial \rho_f}{\partial t} = -\frac{\sigma}{\epsilon}\rho_f}$$

$$\boxed{\rho_f(t) = e^{-\frac{\sigma}{\epsilon}t}\rho_f(0)}$$

for linear homogeneous media

Remark: as $t \to \infty$ we find $\rho_f \to 0$. $\tau \equiv \epsilon/\sigma$ very short typically.

For $t \gg \tau = \epsilon/\sigma$ we face $\rho_f = 0$ thus

$$\boxed{\begin{aligned} &(i.)\ \nabla \cdot \vec{E} = 0 \\ &(ii.)\ \nabla \cdot \vec{B} = 0 \\ &(iii.)\ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ &(iv.)\ \nabla \times \vec{B} = \mu \epsilon \frac{\partial \vec{E}}{\partial t} + \mu \sigma \vec{E} \end{aligned}}$$

(STEADY STATE) MAXWELL'S EQUATIONS IN CONDUCTOR with conductivity σ and permitivity ϵ permeability μ

Applying the identity $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$ we derive modified wave equations:

$$\boxed{\begin{aligned} \nabla^2 \vec{E} &= \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu \sigma \frac{\partial \vec{E}}{\partial t} \\ \nabla^2 \vec{B} &= \mu \epsilon \frac{\partial^2 \vec{B}}{\partial t^2} + \mu \sigma \frac{\partial \vec{B}}{\partial t} \end{aligned}}$$

The complexification of the above allow solutions

$$\boxed{\begin{aligned} \tilde{\vec{E}}(z,t) &= \tilde{\vec{E}}_0 e^{i(\tilde{k} z - \omega t)} \\ \tilde{\vec{B}}(z,t) &= \tilde{\vec{B}}_0 e^{i(\tilde{k} z - \omega t)} \end{aligned}}$$

where $\tilde{k}$ must solve

$$\left.\begin{aligned} \tilde{k}^2 &= \mu \epsilon \omega^2 + i \mu \sigma \omega \\ \tilde{k} &= k + i\kappa \end{aligned}\right\} \text{complex "wave \#"}$$

Griffiths <u>defines</u>

$$k = \omega \sqrt{\frac{\epsilon \mu}{2}} \left[\sqrt{1 + \left(\frac{\sigma}{\epsilon \omega}\right)^2} + 1 \right]^{1/2}$$

$$\kappa = \omega \sqrt{\frac{\epsilon \mu}{2}} \left[\sqrt{1 + \left(\frac{\sigma}{\epsilon \omega}\right)^2} - 1 \right]^{1/2}$$

Curious, what's deal with k and κ

$$k^2 + \kappa^2 = \omega^2\left(\frac{\epsilon\mu}{2}\right)\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1+\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right]$$

$$= \omega^2\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}$$

$$= \omega^2\sqrt{(\epsilon\mu)^2\left[1+\frac{\sigma^2}{\epsilon^2\omega^2}\right]}$$

$$= \sqrt{\omega^4\left[\epsilon^2\mu^2+\frac{\sigma^2\mu^2}{\omega^2}\right]}$$

$$= \sqrt{\omega^2\left[\epsilon^2\mu^2\omega^2+\sigma^2\mu^2\right]}$$

$$= \omega\mu\sqrt{\epsilon^2\omega^2+\sigma^2}$$

Recall $\tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega = \omega\mu(\epsilon\omega + i\sigma)$

$$\boxed{\begin{aligned}\tilde{\vec{E}}(z,t) &= \tilde{\vec{E}}_0\, e^{-\kappa z} e^{i(kz-\omega t)} \\ \tilde{\vec{B}}(z,t) &= \tilde{\vec{B}}_0\, e^{-\kappa z} e^{i(kz-\omega t)}\end{aligned}}$$

$\boxed{\text{Def}^{\underline{n}}\text{/ skin depth } d = 1/\kappa}$

<u>Remark</u>: the wave attenuates to zero inside conductor if we go a few multiples of d into the metal.

$$\boxed{\lambda = \frac{2\pi}{k}, \quad v = \frac{\omega}{k}, \quad n = \frac{ck}{\omega}}$$

governed by the <u>real</u> part of $\tilde{k}$

Maxwell's Equations place further constraints on our attenuated plane wave solution. Much like our earlier calculations for the plane wave case,

$$\nabla \cdot \vec{E} = 0 \implies (\tilde{\vec{E}}_0)_z = 0$$
$$\nabla \cdot \vec{B} = 0 \implies (\tilde{\vec{B}}_0)_z = 0$$

Once more we can set polarization in $\hat{x}$-direction,

Faraday's Law

$$\boxed{\begin{aligned} \tilde{E}(z,t) &= \tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)}\,\hat{x} \\ \tilde{B}(z,t) &= \frac{\tilde{k}}{\omega}\tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)}\,\hat{y} \end{aligned}} \quad - (*)$$

Introduce phase ϕ and modulus K for $\tilde{k} = Ke^{i\phi}$

$$K = |\tilde{k}| = \sqrt{k^2+\kappa^2} = \omega\sqrt{\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}}$$

$$\phi = \tan^{-1}(\kappa/k)$$

Then from $*$ get $\underbrace{B_0 e^{i\delta_B} = \frac{Ke^{i\phi}}{\omega}E_0 e^{i\delta_E}}$

$$\boxed{\frac{B_0}{E_0} = \frac{K}{\omega}}$$

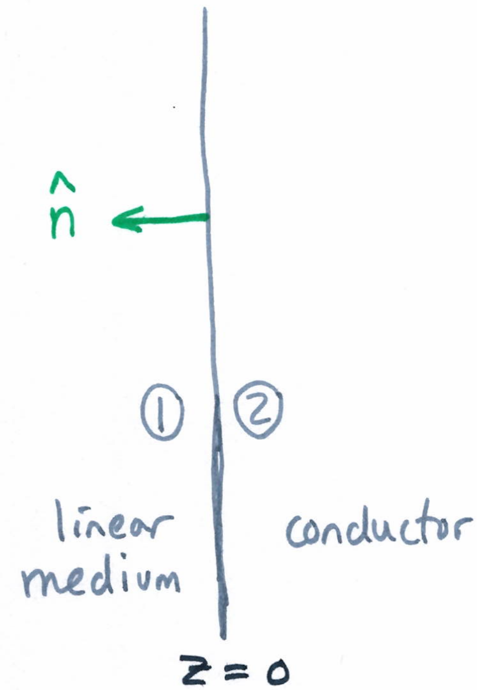
$$\delta_B - \delta_E = \phi$$

the magnetic field lags behind the electric field (see p. 421 for picture)

$$\boxed{\begin{aligned} \vec{E}(z,t) &= E_0 e^{-\kappa z}\cos(kz-\omega t+\delta_E)\,\hat{x} \\ \vec{B}(z,t) &= B_0 e^{-\kappa z}\cos(kz-\omega t+\delta_E+\phi)\,\hat{y} \end{aligned}}$$

Reflection at Conducting Surface

Since free charge is present at interface we face following boundary conditions between media ① and ②,



$$\text{(i.)} \quad \epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f$$

$$\text{(ii.)} \quad B_1^{\perp} - B_2^{\perp} = 0$$

$$\text{(iii)} \quad E_1^{\parallel} - E_2^{\parallel} = 0$$

$$\text{(iv.)} \quad \frac{1}{\mu_1} B_1^{\parallel} - \frac{1}{\mu_2} B_2^{\parallel} = \vec{K}_f \times \hat{n}$$

Consider incident wave as well as reflected and transmitted wave at $z=0$

$$\left.\boxed{\begin{aligned} \tilde{E}_I(z,t) &= \tilde{E}_{0_I} e^{i(k_1 z - \omega t)}\,\hat{x} \\ \tilde{B}_I(z,t) &= \frac{1}{v_1}\tilde{E}_{0_I} e^{i(k_1 z - \omega t)}\,\hat{y} \end{aligned}}\right\} \text{incident wave}$$

$$\left.\boxed{\begin{aligned} \tilde{E}_R(z,t) &= \tilde{E}_{0_R} e^{i(-k_1 z - \omega t)}\,\hat{x} \\ \tilde{B}_R(z,t) &= -\frac{1}{v_1}\tilde{E}_{0_R} e^{i(-k_1 z - \omega t)}\,\hat{y} \end{aligned}}\right\} \text{reflected wave}$$

$$\left.\boxed{\begin{aligned} \tilde{E}_T(z,t) &= \tilde{E}_{0_T} e^{i(\tilde{k}_2 z - \omega t)}\,\hat{x} \\ \tilde{B}_T(z,t) &= \frac{\tilde{k}_2}{\omega}\tilde{E}_{0_T} e^{i(\tilde{k}_2 z - \omega t)}\,\hat{y} \end{aligned}}\right\} \text{transmitted wave}$$

continuing,

$E^{\perp} = 0$ on both sides $\Rightarrow \sigma_f = 0.$

$B^{\perp} = 0$ thus (ii.) $B_1^{\perp} - B_2^{\perp} = 0$ is good.

Condition (iii) $E_1^{\parallel} - E_2^{\parallel}$ yields (at $z=0$),

$$\boxed{\tilde{E}_{0_I} + \tilde{E}_{0_R} = \tilde{E}_{0_T}}$$

Then from (iv.) with $\vec{K}_f = 0$ get $\frac{1}{\mu_1} B_1^{\parallel} = \frac{1}{\mu_2} B_2^{\parallel}$

$$\frac{1}{\mu_1 v_1}\left(\tilde{E}_{0_I} - \tilde{E}_{0_R}\right) = \frac{\tilde{k}_2}{\mu_2 \omega} \tilde{E}_{0_T}$$

$$\Rightarrow \boxed{\tilde{E}_{0_I} - \tilde{E}_{0_R} = \tilde{\beta}\, \tilde{E}_{0_T}} \qquad \tilde{\beta} = \frac{\mu_1 v_1}{\mu_2 \omega} \tilde{k}_2$$

Therefore, in close analogy to our study of refraction,

$$\boxed{\tilde{E}_{0_R} = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right) \tilde{E}_{0_I} \quad \& \quad \tilde{E}_{0_T} = \left(\frac{2}{1+\tilde{\beta}}\right) E_{0_I}}$$

<u>But</u> $\tilde{\beta}$ is <u>complex</u>

PERFECT CONDUCTOR $\longrightarrow \sigma = \infty \longrightarrow k_2 = \infty$

$\Rightarrow \tilde{\beta}$ infinite

$$\therefore \boxed{\tilde{E}_{0_R} = -\tilde{E}_{0_I} \quad \& \quad \tilde{E}_{0_T} = 0}$$

a.k.a. a <u>mirror</u>