LECTURE 30 : ON THE CALCULATION & PROPERTIES OF GRAD, CURL AND DIV

pgs. 329 - 332 in 2020 notes.

7.2 grad, curl and div

Del or "nabla"

In this section we investigate a few natural derivatives we can construct with the operator ∇. Later we will explain what these derivatives *mean.* First, the computation:

Definition 7.2.1.

Suppose f is a scalar function on $\mathbb{R}^3$ then we defined the gradient vector field

$$grad(f) = \nabla f = \langle \partial_x f, \partial_y f, \partial_z f \rangle$$

We studied this before, recall that we can compactly express this by

$$\nabla f = \sum_{i=1}^{3} (\partial_i f)\widehat{x}_i$$

where $\partial_i = \partial/\partial x_i$ and $x_1 = x, x_2 = y$ and $x_3 = z$. Moreover, we have also shown previously in notes or homework that the gradient has the following important properties:

$$\nabla(f+g) = \nabla f + \nabla g, \quad \& \quad \nabla(cf) = c\nabla f, \quad \& \quad \nabla(fg) = (\nabla f)g + f(\nabla g)$$

Together these say that ∇ is a *derivation* of differentiable functions on $\mathbb{R}^n$.

Definition 7.2.2.

$\vec{F} = \langle P, Q, R \rangle$

Suppose $\vec{F} = \langle F_1, F_2, F_3 \rangle$ is a vector field. We define:

$$Div(\vec{F}) = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}. = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

More compactly, we can express the divergence by

$$\nabla \cdot \vec{F} = \sum_{i=1}^{3} \partial_i F_i.$$

$\nabla \cdot (\vec{F} \times \vec{G}) = ?$

You can prove that the divergence satisfies the following important properties:

$$\nabla \cdot (\vec{F} + \vec{G}) = \nabla \cdot \vec{F} + \nabla \cdot \vec{G} \quad \& \quad \nabla \cdot (c\vec{F}) = c\nabla \cdot \vec{F}, \quad \& \quad \nabla \cdot (f\vec{G}) = \nabla f \cdot \vec{G} + f\nabla \cdot \vec{G}$$

$$\nabla \cdot (f\vec{G}) = \sum_{i=1}^{3} \partial_i (f\vec{G})_i = \sum_{i=1} \partial_i (fG_i) = \sum [(\partial_i f) G_i + f \partial_i G_i] = \nabla f \cdot \vec{G} + f \nabla \cdot \vec{G}$$

For example,

$$\nabla \cdot (\vec{F} + c\vec{G}) = \sum_{i=1}^{3} \partial_i (F_i + cG_i) = \sum_{i=1}^{3} \partial_i F_i + c\sum_{i=1}^{3} \partial_i G_i = \nabla \cdot \vec{F} + c\nabla \cdot \vec{G}.$$

Linearity of the divergence follows naturally from linearity of the partial derivatives.

Definition 7.2.3.

Suppose $\vec{F} = \langle F_1, F_2, F_3 \rangle$ is a vector field. We define:

$$Curl(\vec{F}) = \nabla \times \vec{F} = \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle$$

$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$

$\epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1$

More compactly, using the antisymmetric symbol ϵ_{ijk}[4],

$$\nabla \times \vec{F} = \sum_{i,j,k=1}^{3} \epsilon_{ijk}(\partial_i F_j)\widehat{x}_k.$$

You can prove that the curl satisfies the following important properties:

$$\nabla \times (\vec{F} + \vec{G}) = \nabla \times \vec{F} + \nabla \times \vec{G} \quad \& \quad \nabla \times (c\vec{F}) = c\nabla \times \vec{F}, \quad \& \quad \nabla \times (f\vec{G}) = \nabla f \times \vec{G} + f\nabla \times \vec{G}.$$

For example,

$$\begin{aligned} \nabla \times (\vec{F} + c\vec{G}) &= \sum_{i,j,k=1}^{3} \epsilon_{ijk}\partial_i(F_j + cG_j)\widehat{x}_k \\ &= \sum_{i,j,k=1}^{3} \epsilon_{ijk}(\partial_i F_j)\widehat{x}_k + c\sum_{i,j,k=1}^{3} \epsilon_{ijk}(\partial_i G_j)\widehat{x}_k \\ &= \nabla \times \vec{F} + c\nabla \times \vec{G}. \end{aligned}$$

Linearity of the curl follows naturally from linearity of the partial derivatives.

$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$

$\nabla \cdot \vec{A} \neq \vec{A} \cdot \nabla = A_1 \frac{\partial}{\partial x} + A_2 \frac{\partial}{\partial y} + A_3 \frac{\partial}{\partial z}$

It is fascinating how many of the properties of ordinary differentiation generalize to the case of vector calculus. The main difference is that we now must take more care to not commute things that don't commute or confuse functions with vector fields. For example, while it is certainly true that $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$ it is not even sensible to ask the question does $\nabla \cdot \vec{A} = \vec{A} \cdot \nabla$? Notice $\nabla \cdot \vec{A}$ is a function while $\vec{A} \cdot \nabla$ is an operator, apples and oranges.

The proposition below lists a few less basic identities which are at times useful for differential vector calculus.

Proposition 7.2.4.

Let f, g, h be real valued functions on $\mathbb{R}$ and $\vec{F}, \vec{G}, \vec{H}$ be vector fields on $\mathbb{R}$ then (assuming all the partials are well defined)

(i.) $\nabla \cdot (\vec{F} \times \vec{G}) = \vec{G} \cdot (\nabla \times \vec{F}) - \vec{F} \cdot (\nabla \times \vec{G})$
(ii.) $\nabla(\vec{F} \cdot \vec{G}) = \vec{F} \times (\nabla \times \vec{G}) + \vec{G} \times (\nabla \times \vec{F}) + (\vec{F} \cdot \nabla)\vec{G} + (\vec{G} \cdot \nabla)\vec{F}$
(iii.) $\nabla(\vec{F} \times \vec{G}) = (\vec{G} \cdot \nabla)\vec{F} - (\vec{F} \cdot \nabla)\vec{G} + \vec{F}(\nabla \cdot \vec{G}) - \vec{G}(\nabla \cdot \vec{F})$

Proof: Consider (i.), let $\vec{F} = \sum F_i e_i$ and $\vec{G} = \sum G_i e_i$ as usual,

$\epsilon_{ijk} = -\epsilon_{ikj}$

$$\begin{aligned} \nabla \cdot (\vec{F} \times \vec{G}) &= \sum \partial_k[(\vec{F} \times \vec{G})_k] \\ &= \sum \partial_k[\epsilon_{ijk} F_i G_j] \\ &= \sum \epsilon_{ijk}[(\partial_k F_i)G_j + F_i(\partial_k G_j)] \\ &= \sum \epsilon_{ijk}(\partial_i F_j)G_k + -F_j\epsilon_{ikj}(\partial_i G_k) \qquad (7.1) \\ &= \sum G_k(\nabla \times \vec{F})_k - F_j(\nabla \times \vec{G})_j \\ &= \vec{G} \cdot (\nabla \times \vec{F}) - \vec{F} \cdot (\nabla \times \vec{G}). \end{aligned}$$

$\underbrace{F_i \, \epsilon_{ikj} \, \partial_k G_j}_{(\vec{F})_i (\nabla \times \vec{G})_i}$

[4] recall we used this before to better denote harder calculations involving the cross-product

where the sums above are taken over the indices which are repeated in the given expressions. In physics the $\sum$ is often removed and the *einstein index convention* or *implicit summation convention* is used to free the calculation of cumbersome summation symbols. The proof of the other parts of this proposition can be handled similarly, although parts (viii) and (ix) require some thought so I may let you do those for homework[5]. $\square$

Proposition 7.2.5.

If f is a differentiable $\mathbb{R}$-valued function and $\vec{F}$ is a differentiable vector field then

$$(i.)\ \nabla \cdot (\nabla \times \vec{F}) = 0$$
$$(ii.)\ \nabla \times \nabla f = 0$$
$$(iii.)\ \nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$$

± Laplacian

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Before the proof, let me briefly indicate the importance of $(iii.)$ to physics. We learn that in the absence of charge and current the electric and magnetic fields are solutions of

aether

$$\nabla \bullet \vec{E} = 0, \qquad \nabla \times \vec{E} = -\partial_t \vec{B}, \qquad \nabla \bullet \vec{B} = 0, \qquad \nabla \times \vec{B} = \mu_o \epsilon_o \partial_t \vec{E}$$

If we consider the curl of the curl equations we derive,

wave equation

$$\nabla \times (\nabla \times \vec{E}) = \nabla \times (-\partial_t \vec{B}) \ \Rightarrow\ \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\partial_t(\nabla \times \vec{B}) \ \Rightarrow\ \nabla^2 \vec{E} = \mu_o \epsilon_o \partial_t^2 \vec{E}.$$

$$\nabla \times (\nabla \times \vec{B}) = \nabla \times (\mu_o \epsilon_o \partial_t \vec{E}) \ \Rightarrow\ \nabla(\nabla \cdot \vec{B}) - \nabla^2 \vec{B} = \mu_o \epsilon_o \partial_t(\nabla \times \vec{E}) \ \Rightarrow\ \nabla^2 \vec{B} = \mu_o \epsilon_o \partial_t^2 \vec{B}.$$

$V = \frac{1}{\sqrt{\mu_o \epsilon_o}} = 3 \times 10^8$ m/s

These are **wave equations.** If you study the physics of waves you might recognize that the speed of the waves above is $v = 1/\sqrt{\mu_o \epsilon_o}$. This is the speed of light. We have shown that the speed of light apparently depends only on the basic properties of space itself. It is indpendent of the x, y, z coordinates so far as we can see in the usual formalism of electromagnetism. This math was only possible because Maxwell added a term called the displacement current in about 1860. Not many years later radio and TV was invented and all because we knew to look for the possiblility thanks to this mathematics. That said, the notation used above was not common in Maxwell's time. His original presentation of what we now call Maxwell's Equations was given in terms of 20 scalar partial differential equations. Now we enjoy the clarity and precision of the vector formalism. You might be interested to know that Maxwell (like many of the greatest 19-th century physicists) was a Christian. Like Newton, they viewed their enterprise as revealing God's general revelation. Certainly their goal was not to remove God from the picture. They understood that the existence of physical law does not relegate God to non-existence. Rather, we just get a clearer picture on how He created the world in which we live. Just a thought. I have a friend who used to wear a shirt with Maxwell's equations and a taunt "let there be light", when he first wore it he thought he was cleverly debunking God by showing these equations removed the need for God. Now, after accepting Christ, he still wore the shirt but the equations don't mean the same to him any longer. The equations are evidence of God rather than his god.

Proof: I like to use parts (i.) and (ii.) for test questions at times. They're pretty easy, I leave them to the reader. The proof of (iii.) is a bit deeper. We need the well-known identity

[5] relax fall 2011 students this did not happen to you

$\sum_{j=1}^{3} \epsilon_{ikj}\epsilon_{lmj} = \delta_{il}\delta_{km} - \delta_{kl}\delta_{im}$[6]

$$
\begin{aligned}
\nabla \times (\nabla \times \vec{F}) &= \sum_{i,j,k=1}^{3} \epsilon_{ijk}\partial_i(\nabla \times \vec{F})_j \widehat{x}_k \\
&= \sum_{i,j,k=1}^{3} \epsilon_{ijk}\partial_i\Big(\sum_{l,m=1}^{3} \epsilon_{lmj}\partial_l F_m\Big)\widehat{x}_k \\
&= \sum_{i,j,k,l,m=1}^{3} \epsilon_{ijk}\epsilon_{lmj}(\partial_i\partial_l F_m)\widehat{x}_k \\
&= \sum_{i,j,k,l,m=1}^{3} -\epsilon_{ikj}\epsilon_{lmj}(\partial_i\partial_l F_m)\widehat{x}_k \\
&= \sum_{i,k,l,m=1}^{3} (-\delta_{il}\delta_{km} + \delta_{kl}\delta_{im})(\partial_i\partial_l F_m)\widehat{x}_k \\
&= \sum_{i,k,l,m=1}^{3} (-\delta_{il}\delta_{km}\partial_i\partial_l F_m)\widehat{x}_k + \sum_{i,k,l,m=1}^{3} (\delta_{kl}\delta_{im}\partial_i\partial_l F_m)\widehat{x}_k \\
&= \sum_{i,k=1}^{3} -\partial_i\partial_i(F_k\widehat{x}_k) + \sum_{i,k=1}^{3} (\partial_i\partial_k F_i)\widehat{x}_k \\
&= -\sum_{i=1}^{3} \partial_i\partial_i\Big(\sum_{k=1}^{3} F_k\widehat{x}_k\Big) + \sum_{k=1}^{3} \partial_k\Big(\sum_{i=1}^{3} \partial_i F_i\Big)\widehat{x}_k \\
&= -\nabla^2\vec{F} + \nabla(\nabla \cdot \vec{F}). \qquad \square
\end{aligned}
$$

Ex 1.)

$$
\begin{aligned}
\nabla \cdot \langle x^2, y^3, z^4\rangle &= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^4) \\
&= \underline{2x + 3y^2 + 4z^3}.
\end{aligned}
$$

Ex 2.)

$$
\begin{aligned}
\nabla \times \langle y^2, z, 3\rangle &= \det\begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ y^2 & z & 3 \end{bmatrix} \\
&= \langle \partial_y(3) - \partial_z(z), \partial_z(y^2) - \partial_x(3), \partial_x(z) - \partial_y(y^2) \rangle \\
&= \underline{\langle -1, 0, -2y\rangle}.
\end{aligned}
$$

[6] this is actually just the first in a whole sequence of such identities linking the antisymmetric symbol and the kronecker deltas... ask me in advanced calculus, I'll show you the secret formulas