LECTURE 31 : ELECTROMAGNETIC WAVES IN CAVITIES

A hollow pipe or <u>wave guide</u> allows electromagnetic waves to propagate within its interior. We assume a perfect conductor so $\vec{E} = 0$ and $\vec{B} = 0$ within the conducting material. Recall surface charge and currents are induced to force triviality of $\vec{E}$ & $\vec{B}$. This is a limiting principle, as $\sigma \longrightarrow \infty$ the skin-depth goes to zero and all charge/current falls on the edge of the conductor.

$$\boxed{\begin{aligned} \tilde{E}(x,y,z,t) &= \tilde{E}_0(x,y)\, e^{i(kz-\omega t)} \\ \tilde{B}(x,y,z,t) &= \tilde{B}_0(x,y)\, e^{i(kz-\omega t)} \end{aligned}}$$

monochromatic waves within the wave guide (here k real)

<u>MAXWELL'S</u> <u>EQUATIONS</u> : note $\mu_0 \epsilon_0 = \frac{1}{c^2}$ hence $\rightarrow$

$$\text{(i.)}\quad \nabla \cdot \vec{E} = 0 \qquad\qquad \text{(iii.)}\quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\text{(ii.)}\quad \nabla \cdot \vec{B} = 0 \qquad\qquad \text{(iv.)}\quad \nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

The waves need to solve the boundary conditions

$$\text{(i.)}\quad E^{\parallel} = 0 \qquad\qquad \text{(ii.)}\quad B^{\perp} = 0$$

GRIFFITH'S 5^{th} Edition uses notation for complex amplitudes

$$\tilde{E}_0 = E_x\,\hat{x} + E_y\,\hat{y} + E_z\,\hat{z}$$

$$\tilde{B}_0 = B_x\,\hat{x} + B_y\,\hat{y} + B_z\,\hat{z}$$

Consequently, using $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ and $\nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$

$$① \quad \nabla \times \left(\tilde{E}_o e^{i(kz-\omega t)} \right) = -\frac{\partial}{\partial t}\left[\tilde{B}_o e^{i(kz-\omega t)} \right]$$

$$② \quad \nabla \times \left(\tilde{B}_o e^{i(kz-\omega t)} \right) = \frac{1}{c^2}\frac{\partial}{\partial t}\left[\tilde{E}_o e^{i(kz-\omega t)} \right]$$

Notice $\tilde{E}_o$ and $\tilde{B}_o$ can dependent on x, y. Hence the components from $\tilde{E}_o = E_x \hat{x} + E_y \hat{y} + E_z \hat{z}$ etc appear nontrivially in the calculus,

$$①_x: \quad \partial_y \tilde{E}_z - \partial_z \tilde{E}_y = -\frac{\partial \tilde{B}_x}{\partial t} \qquad \underline{\theta = kz - \omega t}$$

$$\partial_y (E_z e^{i\theta}) - \partial_z (E_y e^{i\theta}) = -\frac{\partial}{\partial t}\left[B_x e^{i\theta} \right]$$

$$\frac{\partial E_z}{\partial y} e^{i\theta} - E_y e^{i\theta}(ik) = -B_x e^{i\theta}(-i\omega)$$

$$\boxed{\frac{\partial E_z}{\partial y} - ikE_y = i\omega B_x}$$

$$①_y: \quad \partial_z \tilde{E}_x - \partial_x \tilde{E}_z = -\frac{\partial}{\partial t}\left[\tilde{B}_y \right]$$

$$\frac{\partial}{\partial z}\left[E_x e^{i\theta} \right] - \frac{\partial}{\partial x}\left[E_z e^{i\theta} \right] = -\frac{\partial}{\partial t}\left[B_x e^{i\theta} \right]$$

$$E_x e^{i\theta}(ik) - \frac{\partial E_z}{\partial x} e^{i\theta} = -B_x (-i\omega e^{i\theta})$$

$$\boxed{ikE_x - \frac{\partial E_z}{\partial x} = i\omega B_x}$$

We find (by calculation similar to last page)

$$\left.\begin{aligned}
\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= i\omega B_z \\
\frac{\partial E_z}{\partial y} - ik E_y &= i\omega B_x \\
ik E_x - \frac{\partial E_z}{\partial x} &= i\omega B_y
\end{aligned}\right\}$$

from ① on last page (Faraday's Law)

$$\left.\begin{aligned}
\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} &= -\frac{i\omega}{c^2} E_z \\
\frac{\partial B_z}{\partial y} - ik B_y &= -\frac{i\omega}{c^2} E_x \\
ik B_x - \frac{\partial B_z}{\partial x} &= -\frac{i\omega}{c^2} E_y
\end{aligned}\right\}$$

from ② on last page (Maxwell's Correction)

Solve for E_x, B_x, E_y, B_y

$$E_z = \frac{-c^2}{i\omega}\left[\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right]$$

$$B_z = \frac{1}{i\omega}\left[\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right]$$

$$E_x = \frac{1}{ik}\left[i\omega B_y + \frac{\partial E_z}{\partial x}\right] = \frac{\omega}{k} B_y + \frac{1}{ik}\frac{\partial E_z}{\partial x}$$

$$B_x = \frac{1}{ik}\left[\frac{-i\omega}{c^2} E_y + \frac{\partial B_z}{\partial x}\right] = \frac{-\omega}{kc^2} E_y + \frac{1}{ik}\frac{\partial B_z}{\partial x}$$

$$B_x = \frac{1}{i\omega}\left[-ik E_y + \frac{\partial E_z}{\partial y}\right] = \frac{-k}{\omega} E_y + \frac{1}{i\omega}\frac{\partial E_z}{\partial y}$$

$$E_y\left(\frac{k}{\omega} - \frac{\omega}{kc^2}\right) = \frac{1}{i\omega}\frac{\partial E_z}{\partial y} - \frac{1}{ik}\frac{\partial B_z}{\partial x}$$

Continuing,

$$E_y\left(\frac{k}{\omega}-\frac{\omega}{kc^2}\right)=\frac{1}{i\omega}\frac{\partial E_z}{\partial y}-\frac{1}{ik}\frac{\partial B_z}{\partial x}$$

$$E_y=\left(\frac{1}{\frac{k}{\omega}-\frac{\omega}{kc^2}}\right)\left[\frac{-i\omega}{\omega^2}\frac{\partial E_z}{\partial y}+\frac{ik}{k^2}\frac{\partial B_z}{\partial x}\right]$$

$$E_y=\frac{-i\omega}{k\omega-\omega^3/kc^2}\frac{\partial E_z}{\partial y}+\frac{ik}{k^3/\omega-\frac{\omega k}{c^2}}\frac{\partial B_z}{\partial x}$$

$$E_y=\frac{-ik}{k^2-\omega^2/c^2}\frac{\partial E_z}{\partial y}+\frac{i\omega}{k^2-\omega^2/c^2}\frac{\partial B_z}{\partial x}$$

$$\boxed{E_y=\frac{i}{(\omega/c)^2-k^2}\left(k\frac{\partial E_z}{\partial y}-\omega\frac{\partial B_z}{\partial x}\right)}$$

Similar algebra yields

$$\boxed{\begin{aligned}E_x&=\frac{i}{(\omega/c)^2-k^2}\left[k\frac{\partial E_z}{\partial x}+\omega\frac{\partial B_z}{\partial y}\right]\\ B_x&=\frac{i}{(\omega/c)^2-k^2}\left[k\frac{\partial B_z}{\partial x}-\frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right]\\ B_y&=\frac{i}{(\omega/c)^2-k^2}\left[k\frac{\partial B_z}{\partial y}+\frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\right]\end{aligned}}$$

If we can find the <u>longitudinal</u> <u>components</u> B_z, E_z then E_x, B_x, E_y, B_y can be derived from the equations above.

From $\nabla \cdot \vec{E} = 0$ and $\nabla \cdot \vec{B} = 0$ we will find (Problem 9.28 b)

$$\boxed{\begin{aligned} &(i.)\quad \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c}\right)^2 - k^2\right] E_z = 0 \\ &(ii.)\quad \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c}\right)^2 - k^2\right] B_z = 0 \end{aligned}}$$

The equations above involve E_z alone or B_z alone, these are nice to solve. (uncoupled)

Def^n/ ① If $E_z = 0$ then the wave is called a Transverce Electric wave (TE)

② If $B_z = 0$ then the wave is called a Transverce Magnetic wave (TM)

③ If $E_z = B_z = 0$ then the wave is a TEM wave

Th^m/ no TEM waves in hollow wave guide

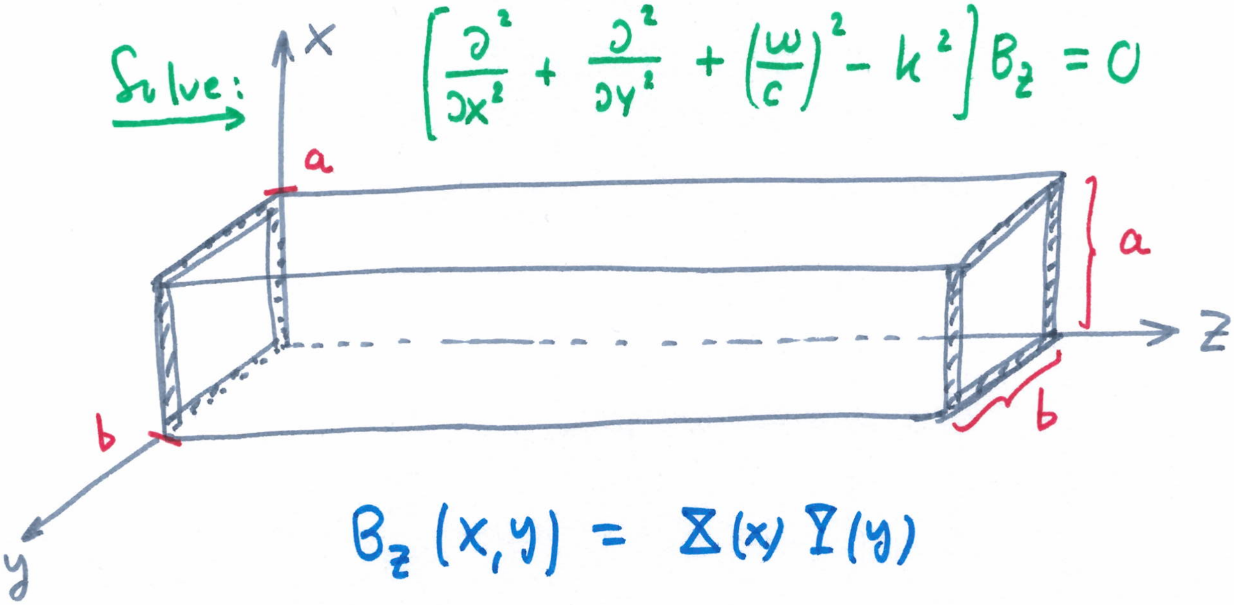
Proof: $E_z = 0 \Rightarrow \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0 \quad (\nabla \cdot \vec{E} = 0)$

$B_z = 0 \Rightarrow \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0 \quad (\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t})$

Thus $\tilde{E}_0 = E_x \hat{x} + E_y \hat{y} + E_z \hat{z}$ has $\nabla \cdot \tilde{E}_0 = 0$ and $\nabla \times \tilde{E}_0 = 0 \Rightarrow \tilde{E}_0 = \nabla^2 \tilde{V}$, but $E^{\parallel} = 0$ implies $\tilde{V}$ must be equipotential on surface $\Rightarrow \tilde{V}$ constant throughout $\therefore \tilde{E}_0 = 0 \Rightarrow$ wave is zero.

TE WAVES IN RECTANGULAR WAVE GUIDE

Solve: $\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c}\right)^2 - k^2\right] B_z = 0$



$$B_z(x,y) = X(x)\,Y(y)$$

$$\underline{Y X'' + X Y'' + \left[(\omega/c)^2 - k^2\right] X Y = 0} \; *$$

Following the usual separation of variables,

(i.) $\boxed{\frac{1}{X}\frac{d^2 X}{dx^2} = -k_x^2}$ & (ii.) $\boxed{\frac{1}{Y}\frac{d^2 Y}{dy^2} = -k_y^2}$

where $\underline{-k_x^2 - k_y^2 + (\omega/c)^2 - k^2 = 0}$ (from $*$)

$$X(x) = A\sin(k_x x) + B\cos(k_x x)$$

However, $X'(0) = 0$ and $X'(a) = 0 \Rightarrow A = 0$

and $B\sin(k_x a) = 0 \therefore \; k_x a = m\pi$

$$\Rightarrow k_x = \frac{m\pi}{a}$$

Continuing, similar arguments for $Y(y)$ imply $k_y = \frac{n\pi}{b}$

$$\boxed{B_z = B_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)}$$

TE_{mn} - mode

(assumes $a \geq b$)

Notice, $-k_x^2 - k_y^2 + (\omega/c)^2 = k^2$ implies

$$k = \sqrt{\left(\frac{\omega}{c}\right)^2 - \pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]}$$

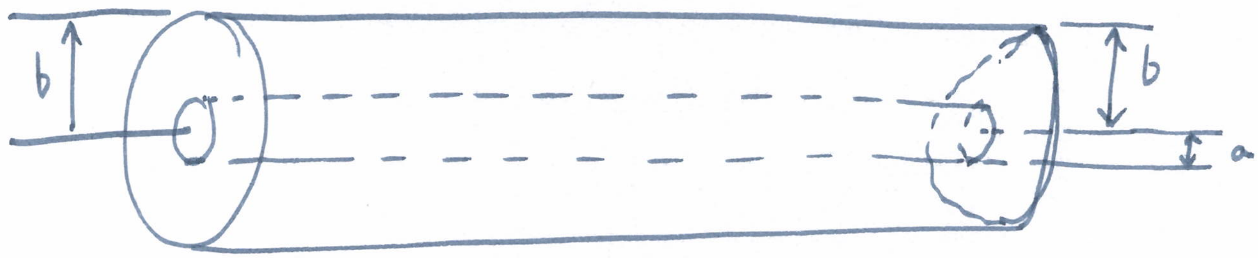
$$\boxed{\text{Def}^{\underline{n}}/ \quad \omega_{mn} = c\pi\sqrt{(m/a)^2 + (n/b)^2}}$$ (CUT-OFF FREQUENCY)

Remark: If $\omega < c\pi\sqrt{(m/a)^2 + (n/b)^2}$ (that is $\omega < \omega_{mn}$) then k is imaginary and such TEM-modes are attenuated. (Hence "CUT-OFF")

$$\omega_{10} = \frac{c\pi}{a}$$ ← lowest frequency that can propagate in the wave guide

Remark: GRIFFITHS DISCUSSES a bit more here about group velocity and wave fronts. I'm silent about that part.

COAXIAL TRANSMISSION LINE



Conductors at $s = a$ and $s = b$ allow for nontrivial solutions to Laplace's Equation (unlike the hollow wave guide thus far considered)

Here $E_z = 0$ and $B_z = 0$ solutions exist and

$$\underbrace{k = \omega/c}_{\text{waves travel at } c} \quad \left(\begin{array}{c}\text{Faraday's \& Maxwell's} \\ \text{corrections}\end{array}\right)$$

Also, $c B_y = E_x$ and $c B_x = -E_y$

equations of electrostatics & magnetostatics for empty space

$$\left\{\begin{array}{ll} \dfrac{\partial E_x}{\partial x} + \dfrac{\partial E_y}{\partial y} = 0 & \dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y} = 0 \\ \dfrac{\partial B_x}{\partial x} + \dfrac{\partial B_y}{\partial y} = 0 & \dfrac{\partial B_y}{\partial x} - \dfrac{\partial B_x}{\partial y} = 0 \end{array}\right]$$

Hence, for infinite line charge & infinite straight current we know solutions are:

$$\vec{E}_0(s, \phi) = \frac{A}{s}\hat{s} \quad \text{and} \quad \vec{B}_0 = \frac{A}{cs}\hat{\phi}$$

$$\boxed{\begin{aligned} \vec{E}(s, \phi, z, t) &= \frac{A}{s}\cos(kz - \omega t)\,\hat{s} \\ \vec{B}(s, \phi, z, t) &= \frac{A}{cs}\cos(kz - \omega t)\,\hat{\phi} \end{aligned}}$$