LECTURE 31 : LINE INTEGRALS

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7.3 line integrals

In this section I describe several integrals over curves. These integrals capture something intrinsic between the given function or vector field and the given curve. For convenience we state the definitions in terms of a particular type of parametrized path, but we prove that the choice of the parameter is independent from the value of the integral. Placing a parameter on a curve is a choice of coordinates and our integrals are coordinate free. For the integral with respect to arclength $\int_C f ds$, or scalar line integral, we are adding up some f scalar function along the curve. On the other hand, the line-integral $\int_C \vec{F} \cdot d\vec{r}$ is taken along an oriented curve and it computes the amount which the vector field $\vec{F}$ points tangentially to the curve. From a physical perspective, the line-integral gives the work done by a force over a curve. This is much more general than the simplisitic constant force or constant direction idea of work you have seen in previous portions of the calculus sequence. [7]. The organization of this section is as follows: we begin with curves and some terminology, the define the integral with respect to arclength, the line integral and finally we investigate connections between the two concepts as well as the application of differential notation as an organizing principle for quick assembly of a line-integral.

7.3.1 curves and paths

Definition 7.3.1.

A **path** in $\mathbb{R}^3$ is a continuous function $\vec{\gamma}$ with connected domain I such that $\vec{\gamma} : I \subset \mathbb{R} \to \mathbb{R}^3$. If $\partial I = \{a, b\}$ then we say that $\vec{\gamma}(a)$ and $\vec{\gamma}(b)$ are the endpoints of the path $\vec{\gamma}$. When $\vec{\gamma}$ has continuous derivatives of all orders we say it is a smooth path (of class C^∞), if it has at least one continuous derivative we say it is a differentiable path(of class C^1). When $I = [a, b]$ then the path is said to go from $\vec{\gamma}(a) = P$ to $\vec{\gamma}(b) = Q$ and the image $C = \vec{\gamma}([a, b])$ is said to be an **oriented curve** C from P to Q. The same curve from Q to P is denoted $-C$. We say C and $-C$ have opposite orientations.

Hopefully most of this is already familar from our earlier work on parametrizations. I give another example just in case.

Example 7.3.2. *The line-segment L from $(1, 2, 3)$ to $(5, 5, 5)$ has parametric equations $x = 1 + 4t, y = 2 + 3t, z = 3 + 2t$ for $0 \leq t \leq 1$. In other words, the path $\vec{\gamma}(t) = \langle 1 + 4t, 2 + 3t, 3 + 2t \rangle$ covers the line-segment L. In constrast $-L$ goes from $(5, 5, 5)$ to $(1, 2, 3)$ and we can parametrize it by $x = 5 - 4u, y = 5 - 3u, z = 5 - 2u$ or in terms of a vector-formula $\vec{\gamma}_{reverse}(u) = \langle 5 - 4u, 5 - 3u, 5 - 2u \rangle$. How are these related? Observe:*

$$\vec{\gamma}_{reverse}(0) = \vec{\gamma}(1) \qquad \& \qquad \vec{\gamma}_{reverse}(1) = \vec{\gamma}(0)$$

Generally, $\vec{\gamma}_{reverse}(t) = \vec{\gamma}(1 - t)$.

We can generalize this construction to other curves. If we are given C from P to Q parametrized by $\vec{\gamma} : [a, b] \to \mathbb{R}^3$ then we can parametrize $-C$ by $\vec{\gamma}_{reverse} : [a, b] \to \mathbb{R}^3$ defined by $\vec{\gamma}_{reverse}(t) = \vec{\gamma}(a + b - t)$. Clearly we have $\vec{\gamma}_{reverse}(a) = \vec{\gamma}(b) = Q$ whereas $\vec{\gamma}_{reverse}(b) = \vec{\gamma}(a) = Q$. Perhaps it is interesting to compare these paths at a common point,

$$\gamma(t) = \vec{\gamma}_{reverse}(a + b - t)$$

[7] recall $W = \vec{F} \cdot \Delta\vec{x}$ from the beginning of this course and $W = \int_a^b F(x)dx$ from calculus I or II

The velocity vectors naturally point in opposite directions, (by the chain-rule)

$$\frac{d\vec{\gamma}}{dt}(t) = -\frac{d\vec{\gamma}_{reverse}}{dt}(a+b-t).$$

Example 7.3.3. *Suppose $\vec{\gamma}(t) = \langle \cos(t), \sin(t) \rangle$ for $\pi \leq t \leq 2\pi$ covers the oriented curve C. If we wish to parametrize $-C$ by $\vec{\beta}$ then we can use*

$$\vec{\beta}(t) = \vec{\gamma}(3\pi - t) = \langle \cos(3\pi - t), \sin(3\pi - t) \rangle$$

Simplifying via trigonometry yields $\vec{\beta}(t) = \langle -\cos(t), -\sin(t) \rangle$ for $\pi \leq t \leq 2\pi$. You can easily verify that $\vec{\beta}$ covers the lower half of the unit-circle in a CW-fashion, it goes from (1.0) to $(-1, 0)$

What I have just described is a general method to reverse a path whilst keeping the same domain for the new path. Naturally, you might want to use a different domain after you change the parametrization of a given curve. Let's settle the general idea with a definition. This definition describes what we allow as a reasonable reparametrization of a curve.

Definition 7.3.4.

Let $\vec{\gamma}_1 : [a_1, b_1] \to \mathbb{R}^3$ be a path. We say another path $\vec{\gamma}_2 : [a_2, b_2] \to \mathbb{R}^3$ is a **reparametrization** of $\vec{\gamma}_1$ if there exists a bijective (one-one and onto), continuous function $u : [a_1, b_1] \to [a_2, b_2]$ with continuous inverse $u^{-1} : [a_2, b_2] \to [a_1, b_1]$ such that $\vec{\gamma}_1(t) = \vec{\gamma}_2(u(t))$ for all $t \in [a_1, b_1]$. If the given curve is smooth or k-times differentiable then we also insist that the transition function u and its inverse be likewise smooth or k-times differentiable.

In short, we want the allowed reparametrizations to capture the same curve without adding any artificial stops, starts or multiple coverings. If the original path wound around a circle 10 times then we insist that the allowed reparametrizations also wind 10 times around the circle. Finally, let's compare the a path and its reparametrization's velocity vectors, by the chain rule we find:

$$\vec{\gamma}_1(t) = \vec{\gamma}_2(u(t)) \qquad \Rightarrow \qquad \frac{d\vec{\gamma}_1}{dt}(t) = \frac{du}{dt}\frac{d\vec{\gamma}_2}{dt}(u(t)).$$

This calculation is important in the section that follows. Observe that:

1. if $du/dt > 0$ then the paths progress in the same direction and are **consitently oriented**
2. if $du/dt < 0$ then the paths go in opposite directions and are **oppositely oriented**

Reparametrizations with $du/dt > 0$ are said to be **orientation preserving**.

7.3.2 line-integral of scalar function

These are also commonly called the **integral with respect to arclength**. In lecture we framed the need for this definition by posing the question of finding the area of a curved fence with height $f(x, y)$. It stood to reason that the infinitesimal area dA of the curved fence over the arclength ds would simply be $dA = f(x, y)ds$. Then integration is used to sum all the little areas up. Moreover, the natural calculation to accomplish this is clearly as given below:

Definition 7.3.5.

Let $\vec{\gamma} : [a,b] \to C \subset \mathbb{R}^n$ be a differentiable path and suppose that $C \subset dom(f)$ for a continuous function $f : dom(f) \to \mathbb{R}$ then the **scalar line integral of f along** C is

$$\int_C f\, ds \equiv \int_a^b f(\vec{\gamma}(t))\, ||\vec{\gamma}\,'(t)||\, dt.$$

We should check to make sure there is no dependence on the choice of parametrization above. If there was then this would not be a reasonable definition. Suppose $\vec{\gamma}_1(t) = \vec{\gamma}_2(u(t))$ for $a_1 \leq t \leq b_1$ where $u : [a_1, b_1] \to [a_2, b_2]$ is differentiable and strictly monotonic. Note

$$\begin{aligned}\int_{a_1}^{b_1} f(\vec{\gamma}_1(t)) \left|\left|\frac{d\vec{\gamma}_1}{dt}\right|\right| dt &= \int_{a_1}^{b_1} f(\vec{\gamma}_2(u(t))) \left|\left|\frac{du}{dt}\frac{d\vec{\gamma}_2}{dt}(u(t))\right|\right| dt \\ &= \int_{a_1}^{b_1} f(\vec{\gamma}_2(u(t))) \left|\left|\frac{d\vec{\gamma}_2}{dt}(u(t))\right|\right| \cdot \left|\frac{du}{dt}\right| dt\end{aligned}$$

If u is orientation preserving then $du/dt > 0$ hence $u(a_1) = a_2$ and $u(b_1) = b_2$ and thus

$$\begin{aligned}\int_{a_1}^{b_1} f(\vec{\gamma}_1(t)) \left|\left|\frac{d\vec{\gamma}_1}{dt}\right|\right| dt &= \int_{a_1}^{b_1} f(\vec{\gamma}_2(u(t))) \left|\left|\frac{d\vec{\gamma}_2}{dt}(u(t))\right|\right| \frac{du}{dt}\, dt \\ &= \int_{a_2}^{b_2} f(\vec{\gamma}_2(u) \left|\left|\frac{d\vec{\gamma}_2}{du}\right|\right| du.\end{aligned}$$

On the other hand, if $du/dt < 0$ then $|du/dt| = -du/dt$ and the bounds flip since $u(a_1) = b_2$ and $u(b_1) = a_2$

$$\begin{aligned}\int_{a_1}^{b_1} f(\vec{\gamma}_1(t)) \left|\left|\frac{d\vec{\gamma}_1}{dt}\right|\right| dt &= -\int_{a_1}^{b_1} f(\vec{\gamma}_2(u(t))) \left|\left|\frac{d\vec{\gamma}_2}{dt}(u(t))\right|\right| \frac{du}{dt}\, dt \\ &= -\int_{b_2}^{a_2} f(\vec{\gamma}_2(u) \left|\left|\frac{d\vec{\gamma}_2}{du}\right|\right| du. \\ &= \int_{a_2}^{b_2} f(\vec{\gamma}_2(u) \left|\left|\frac{d\vec{\gamma}_2}{du}\right|\right| du.\end{aligned}$$

Note, the definition requires me to flip the bounds before I judge if we have the same result. This is implicit in the statement in the definition that $dom(\vec{\gamma}) = [a,b]$ this forces $a < b$ and hence the integral in turn. Technical details aside we have derived the following important fact:

$$\boxed{\int_C f ds = \int_{-C} f ds}$$

The scalar-line integral of function with no attachment to C is indpendent of the orientation of the curve. Given our original motivation for calculating the area of a curved fence this is not surprising.

One convenient notation calculation of the scalar-line integral is given by the dot-notation of Newton. Recall that $dx/dt = \dot{x}$ hence $\vec{\gamma} = \langle x, y, z\rangle$ has $\vec{\gamma}\,'(t) = \langle \dot{x}, \dot{y}, \dot{z}\rangle$. Thus, for a space curve,

$$\int_C f\ ds \equiv \int_a^b f(x,y,z)\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\,dt.$$

We can also calculate the scalar line integral of f along some curve which is made of finitely many differentiable segments, we simply calculate each segment's contribution and sum them together. Just like calculating the integral of a piecewise continuous function with a finite number of jump-discontinuities, you break it into pieces.

Furthermore, notice that if we calculate the scalar line integral of the constant function $f = 1$ then we will obtain the arclength of the curve. More generally the scalar line integral calculates the weighted sum of the values that the function f takes over the curve C. If we divide the result by the length of C then we would have the average of f over C.

Example 7.3.6. *Suppose the linear mass density of a helix $x = R\cos(t), y = R\sin(t), z = t$ is given by $dm/dz = z$. Calculate the total mass around the two twists of the helix given by $0 \le t \le 4\pi$.*

$$\begin{aligned} m_{total\ on\ C} = \int_C z ds &= \int_0^{4\pi} z\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\ dt \\ &= \int_0^{4\pi} t\sqrt{R^2+1}\ dt \\ &= \frac{t^2\sqrt{R^2+1}}{2}\bigg|_0^{4\pi} \\ &= \boxed{8\pi^2\sqrt{R^2+1}.} \end{aligned} \tag{7.2}$$

In constrast to total mass we could find the arclength by simply adding up ds, the total length L of C is given by

$$\begin{aligned} L = \int_C ds &= \int_0^{4\pi} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\ dt \\ &= \int_0^{4\pi} \sqrt{R^2+1}\ dt \\ &= \boxed{4\pi\sqrt{R^2+1}.} \end{aligned}$$

Definition 7.3.7.

Let C be a curve with length L then the average of f over C is given by

$$f_{avg} = \frac{1}{L}\int_C f\ ds.$$

Example 7.3.8. *The average mass per unit length of the helix with $dm/dz = z$ as studied above is given by*

$$m_{avg} = \frac{1}{L}\int_C f\ ds = \frac{1}{4\pi\sqrt{R^2+1}}8\pi^2\sqrt{R^2+1} = \boxed{2\pi.}$$

Since $z = t$ and $0 \le t \le 4\pi$ over C this result is hardly surprising.

Another important application of the scalar line integral is to find the center of mass of a wire. The idea here is nearly the same as we discussed for volumes, the difference is that the mass is distributed over a one-dimensional space so the integration is one-dimensional as opposed to two-dimensional to find the center of mass for a planar laminate or three-dimensional to find the center of mass for a volume.

Definition 7.3.9.

Let C be a curve with length L and suppose $dM/ds = \delta$ is the mass-density of C. The total mass of the curve found by $M = \int_C \delta ds$. The **centroid** or **center of mass** for C is found at $(\bar{x}, \bar{y}, \bar{z})$ where

$$\bar{x} = \frac{1}{M}\int_C x\delta\, ds, \qquad \bar{y} = \frac{1}{M}\int_C y\delta\, ds, \qquad \bar{z} = \frac{1}{M}\int_C z\delta\, ds.$$

Often the centroid is found off the curve.

Example 7.3.10. *Suppose $x = R\cos(t), y = R\sin(t), z = h$ for $0 \leq t \leq \pi$ for a curve with $\delta = 1$. Clearly $ds = Rdt$ and thus $M = \int_C \delta ds = \int_0^\pi Rdt = \pi R$. Consider,*

$$\bar{x} = \frac{1}{\pi R}\int_C xds = \frac{1}{\pi R}\int_0^\pi R^2\cos(t)dt = 0$$

whereas,

$$\bar{y} = \frac{1}{\pi R}\int_C yds = \frac{1}{\pi R}\int_0^\pi R^2\sin(t)dt = \frac{1}{\pi R}(-R^2\cos(t)\Big|_0^\pi = \frac{2R}{\pi}$$

The reader can easily verify that $\bar{z} = h$ hence the centroid is at $(0, \frac{2R}{\pi}, h)$.

Of course there are many other applications, but I believe these should suffice for our current purposes. We will eventually learn that $\int_C \vec{F} \bullet \vec{T} ds$ and $\int_C \vec{F} \bullet \vec{N} ds$ are also of interest, but we should cover other topics before returning to these. Incidentally, it is pretty obvious that we have the following properties for the scalar-line integral:

$$\int_C (f + cg)ds = \int_C fds + c\int_C gds \qquad \& \qquad \int_{C\cup\bar{C}} fds = \int_C fds + \int_{\bar{C}} fds$$

in addition if $f \leq g$ on C then $\int_C fds \leq \int_C gds$. I leave the proof to the reader.

7.3.3 line-integral of vector field

For those of you who know a little physics, the motivation to define this integral follows from our desire to calculate the work done by a variable force $\vec{F}$ on some particle as it traverses C. In particular, we expect the little bit of work dW done by $\vec{F}$ as the particle goes from $\vec{r}$ to $\vec{r} + d\vec{r}$ is given by $\vec{F} \bullet d\vec{r}$. Then, to find total work, we integrate:

Definition 7.3.11.

Let $\vec{\gamma} : [a, b] \to C \subset \mathbb{R}^3$ be a differentiable path which covers the oriented curve C and suppose that $C \subset dom(\vec{F})$ for a continuous vector field $\vec{F}$ on $\mathbb{R}^3$ then the **vector line integral of $\vec{F}$ along** C is denoted and defined as follows:

$$\int_C \vec{F} \bullet d\vec{r} = \int_a^b \vec{F}(\vec{\gamma}(t)) \bullet \frac{d\vec{\gamma}}{dt}\, dt.$$

This integral measures the **work** done by $\vec{F}$ over C. Alternatively, this is also called the **circulation** of $\vec{F}$ along C, however that usage tends to appear in the case that C is a loop. A closed curve is defined to be a curve which has the same starting and ending points. We can indicate the line-integral is taken over a loop by the notation $\oint_C \vec{F} \bullet d\vec{r}$. As with the case of the scalar line integral we ought to examine the dependence of the definition on the choice of parametrization for C. If we were to find a dependence then we would have to modify the definition to make it reasonable. Once more consider the reparametrization $\vec{\gamma}_2$ of $\vec{\gamma}_1$ by a strictly monotonic differentiable function $u : [a_1, b_1] \to [a_2, b_2]$ where we have $\vec{\gamma}_1(t) = \vec{\gamma}_2(u(t))$. Consider,

$$\begin{aligned}
\int_{a_1}^{b_1} \vec{F}(\vec{\gamma}_1(t)) \bullet \frac{d\vec{\gamma}_1}{dt}\, dt &= \int_{a_1}^{b_1} \vec{F}(\vec{\gamma}_2(u(t)))\frac{d}{dt}\Big[\vec{\gamma}_2(u(t))\Big]\, dt \\
&= \int_{a_1}^{b_1} \vec{F}(\vec{\gamma}_2(u(t)))\frac{d\vec{\gamma}_2}{dt}(u(t)) \cdot \frac{du}{dt}\, dt \\
&= \int_{u(a_1)}^{u(b_1)} \vec{F}(\vec{\gamma}_2(u)\frac{d\vec{\gamma}_2}{du}\, du
\end{aligned}$$

If u is orientation preserving then $u(a_1) = a_2$ and $u(b_1) = b_2$ and we find the integral $\int_{a_1}^{b_1} \vec{F}(\vec{\gamma}_1(t)) \bullet \frac{d\vec{\gamma}_1}{dt}\, dt = \int_{a_2}^{b_2} \vec{F}(\vec{\gamma}_2(u)\frac{d\vec{\gamma}_2}{du}\, du$. However, if u is orientation reversing then we find $u(a_1) = b_2$ and $u(b_1) = a_2$ hence $\int_{a_1}^{b_1} \vec{F}(\vec{\gamma}_1(t)) \bullet \frac{d\vec{\gamma}_1}{dt}\, dt = -\int_{b_2}^{a_2} \vec{F}(\vec{\gamma}_2(u)\frac{d\vec{\gamma}_2}{du}\, du$. Therefore, we find that

$$\boxed{\int_{-C} \vec{F} \bullet d\vec{r} = -\int_C \vec{F} \bullet d\vec{r}}$$

assumes f indep. of C

$$\boxed{\int_C f\,ds = \int_{-C} f\,ds}$$

This is actually not surprising if you think about the motivation for the integral. The integral *measures how the vector field $\vec{F}$ points in the same direction as C.* The curve $-C$ goes in the opposite direction thus it follows the sign should differ for the line-integral. Long story short, we must take line-integrals with respect to **oriented curves**.

Is instructive to relate the line-integral and the integral with respect to arclength[8],

$$\int_C \vec{F} \bullet d\vec{r} = \int_a^b \vec{F}(\vec{\gamma}(t)) \bullet \frac{d\vec{\gamma}}{dt}\, dt = \int_a^b \vec{F}(\vec{\gamma}(t)) \bullet \left[\frac{\vec{\gamma}\,'(t)}{||\vec{\gamma}\,'(t)||}\right] ||\vec{\gamma}\,'(t)||\, dt = \int_C (\vec{F} \bullet \vec{T}) ds.$$

T

tangential component of force.

As the last equality indicates, the vector line integral of $\vec{F}$ is given by the scalar line integral of the tangential component $\vec{F} \cdot \vec{T}$. Thus the vector line integral of $\vec{F}$ along C gives us a measure of how much $\vec{F}$ points in the same direction as the oriented curve C. If the vector field always cuts the path perpendicularly (if it was normal to the curve) then the vector line integral would be zero.

Example 7.3.12. *Suppose $\vec{F}(x, y, z) = \langle y, z-1+x, 2-x \rangle$ and suppose C is an ellipse on the plane $z = 1 - x - y$ where $x^2 + y^2 = 4$ and we orient C in the CCW direction relative to the xy-plane with positive normal (imagine pointing your right hand above the xy-plane and your fingers curl around the ellipse in the CCW direction). Our goal is to calculate $\int_C \vec{F} \bullet d\vec{r}$. To do this we must first understand how to parametrize the ellipse:*

$$x = 2\cos(t), \quad y = 2\sin(t)$$

[8] think about this equality with $-C$ in place of C, why is this not a contradiction? On first glance you might think only the lhs is orientation dependent.

gives $x^2 + y^2 = 4$ and the CCW direction. To find z we use the plane equation,

$$z = 1 - x - y = 1 - 2\cos(t) - 2\sin(t)$$

Therefore,

$$\vec{r}(t) = \langle 2\cos(t), 2\sin(t), 1 - 2\cos(t) - 2\sin(t) \rangle$$

thus

$$\frac{d\vec{r}}{dt} = \Big\langle -2\sin(t), 2\cos(t), 2[\sin(t) - \cos(t)] \Big\rangle$$

Evaluate $\vec{F}(x,y,z) = \langle y, z-1+x, 2-x \rangle$ at $x = 2\cos(t)$, $y = 2\sin(t)$ and $z = 1 - 2\cos(t) - 2\sin(t)$ to find

$$\vec{F}(\vec{r}(t)) = \Big\langle 2\sin(t), -2\sin(t), 2[1 - \cos(t)] \Big\rangle$$

Now put it together,

$$\begin{aligned}
\oint_C \vec{F} \bullet d\vec{r} &= \int_0^{2\pi} \Big\langle 2\sin(t), -2\sin(t)), 2[1-\cos(t)] \Big\rangle \bullet \Big\langle -2\sin(t), 2\cos(t), 2[\sin(t) - \cos(t)] \Big\rangle \, dt \\
&= \int_0^{2\pi} \Big[-4\sin^2(t) - 4\sin(t)\cos(t) + 4[1-\cos(t)][\sin(t) - \cos(t)] \Big] \, dt \\
&= \int_0^{2\pi} \Big[-4\sin^2(t) - 4\sin(t)\cos(t) + 4\sin(t) - 4\cos(t) - 4\cos(t)\sin(t) + 4\cos^2(t) \Big] \, dt \\
&= 4\int_0^{2\pi} \big[\cos^2(t) - \sin^2(t)\big] \, dt \\
&= 4\int_0^{2\pi} \big[\tfrac{1}{2}(1+\cos(2t) - \tfrac{1}{2}(1-\cos(2t))\big] \, dt \\
&= 4\int_0^{2\pi} \big[\tfrac{1}{2}(1+\cos(2t) - \tfrac{1}{2}(1-\cos(2t))\big] \, dt \\
&= 4\int_0^{2\pi} \big[\cos(2t)\big] \, dt \\
&= 0.
\end{aligned}$$

The example above indicates how we apply the definition of the line-integral directly. Sometimes it is convenient to use differential notation. If C is parametrized by $\vec{r} = \langle x, y, z \rangle$ for $a \leq t \leq b$ we define the integrals of the differential forms Pdx, Qdy and Rdz in the following way:

Definition 7.3.13.

> Let $\vec{r} : [a,b] \to C \subset \mathbb{R}^3$ be a differentiable path which covers the oriented curve C and suppose that $C \subset dom(\langle P, Q, R \rangle)$ for a continuous vector field $\langle P, Q, R \rangle$ on $\mathbb{R}^3$ then we define
>
> $$\int_C P dx = \int_a^b P(\vec{r}(t)) \frac{dx}{dt} \, dt, \quad \int_C Q dx = \int_a^b Q(\vec{r}(t)) \frac{dy}{dt} \, dt, \quad \int_C R dx = \int_a^b R(\vec{r}(t)) \frac{dz}{dt} \, dt,$$

These are not basic calculations and in and of themselves they are not terribly interesting. I suppose that the $\int_C Pdx$ measures the work done by the x-vector-component of $\vec{F} = \langle P, Q, R \rangle$ whereas the

$\int_C Qdy$ and the $\int_C Rdz$ measure the work done by the y and z vector components of $\vec{F} = \langle P, Q, R\rangle$. Primarily, these are interesting since when we add them we obtain the full line-integral:

$$\boxed{\int_C \langle P, Q, R\rangle \bullet d\vec{r} = \int_C (Pdx + Qdy + Rdz)}$$

I invite the reader to verify the formula above. I will illustrate its use in many examples to follow. It should be emphasized that these are just notation to organize the line integral.

Example 7.3.14. *Calculate $\int_C \vec{F} \bullet d\vec{r}$ for $\vec{F}(x, y, z) = \langle y, z-1+x, 2-x\rangle$ given that C is parametrized by $x = \cos(t), y = \sin(t), z = 1$ for $0 \le t \le 2\pi$. Note that*

$$dx = -\sin(t)dt, \quad dy = \cos(t)dt, \quad dz = 0$$

Thus, for $P = y = \sin(t)$, $Q = z - 1 + x = \cos(t)$ and $R = 2 - x = 2 - \cos(t)$ we find

$$\oint_C \vec{F} \bullet d\vec{r} = \int_C (Pdx + Qdy + Rdz) = \int_0^{2\pi} -\sin^2(t)dt + \cos(t)\cos(t)dt = \int_0^{2\pi} \cos(2t)dt = 0.$$

You might wonder if the integral around a closed curve is always zero.

Example 7.3.15. *Let $\vec{F} = \langle y, -x\rangle$ and suppose $x = R\cos(t), y = R\sin(t)$ parametrizes C for $0 \le t \le 2\pi$. Calculate,*

$$Pdx + Qdy = -yR\sin(t)dt - xR\cos(t)dt = -R^2\sin^2(t)dt - R^2\cos^2(t)dt = -R^2dt$$

Thus, $\oint_C \vec{F} \bullet d\vec{r} = -\int_0^{2\pi} R^2dt = -2\pi R^2$.

Apparently just because we integrate around a loop it does not mean the answer is zero. I suspect that there are loops for which $\vec{F}(x, y, z) = \langle y, z - 1 + x, 2 - x\rangle$ has nonzero circulation. We will return to that example once more in the next section after we learn a test to determine if the $\oint_C \vec{F} \bullet d\vec{r} = 0$ without direct calculation. In conclusion, I should mention that the properties below are easily proved by direct calculation on the defintion,

$$\int_C (\vec{F} + c\vec{G}) \bullet d\vec{r} = \int_C \vec{F} \bullet d\vec{r} + c\int_C \vec{G} \bullet d\vec{r} \qquad \& \qquad \int_{C\cup\bar{C}} \vec{F} \bullet d\vec{r} = \int_C \vec{F} \bullet d\vec{r} + \int_{\bar{C}} \vec{F} \bullet d\vec{r}.$$

Example 1) $\int_{C_1} xy\,dx + (y+z)\,dy + x\,dz$ where

C_1 is given by $x = t,\ y = t^2,\ z = t^3$ for $0 \le t \le 1$

(**Remark:** this is equivalent to calculating $\int_{C_1} \vec{F} \cdot d\vec{r}$
where $\vec{F} = \langle xy,\ y+z,\ x \rangle$)

$x = t \qquad y = t^2 \qquad z = t^3$

$dx = dt \qquad dy = 2t\,dt \qquad dz = 3t^2\,dt$

Then,

$$\int_{C_1} xy\,dx + (y+z)\,dy + x\,dz = \int_0^1 t\,t^2\,dt + (t^2+t^3)\,2t\,dt + t\,(3t^2\,dt)$$

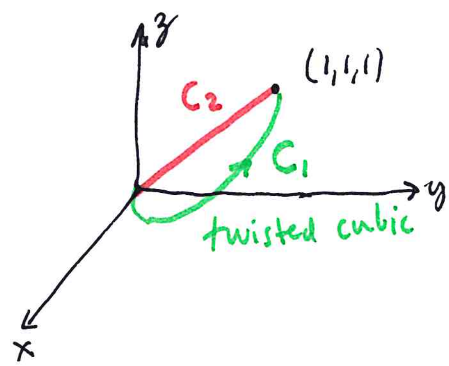
$$= \int_0^1 \left[t^3 + 2t^3 + 2t^4 + 3t^3\right] dt$$

$$= \int_0^1 \left[7t^3 + 2t^4\right] dt$$

$$= \frac{7}{4} + \frac{2}{5}$$

$$= \frac{35+8}{20}$$

$$= \boxed{\frac{43}{20}}$$



Example 2 same $\vec{F}$ as last example, but C_2 = line segment from $(0,0,0)$ to $(1,1,1)$
then C_2 : $x = t,\ y = t,\ z = t \quad 0 \le t \le 1$ hence $dx = dy = dz = dt$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^1 t^2\,dt + (t+t)\,dt + t\,dt$$

$$= \int_0^1 (t^2 + 3t)\,dt$$

$$= \frac{1}{3} + \frac{3}{2}$$

$$= \frac{2+9}{6} = \boxed{\frac{11}{6}}$$