7.4   conservative vector fields

In this section we discuss how to identify a conservative vector field and how to use it. There are about 5 equivalent ideas and our job in this section is to explore how these concepts are connected. We also make a few connections with physics and it should be noted that part of the terminology is borrowed from classical mechanics. Let us begin with the fundamental theorem for line integrals.
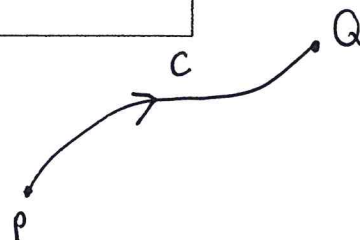
Theorem 7.4.1. *Fundamental Theorem of Calculus for Line Integrals.*

> Suppose f is differentiable on some open set containing the oriented curve C from P to Q then
> $$\int_C \nabla f \cdot d\vec{r} = f(Q) - f(P).$$

Proof: let $\vec{r} : [a,b] \to C \subset \mathbb{R}^n$ parametrize C and calculate:

$$\int_C \nabla f \cdot d\vec{r} = \int_a^b \nabla f(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} \, dt$$

$$= \int_a^b \frac{d}{dt}\left[f(\vec{r}(t)) \right] dt$$

$$= f(\vec{r}(b)) - f(\vec{r}(a))$$

$$= f(Q) - f(P).$$

The two critical steps above are the application of the multivariate chain-rule and then in the next to last step we apply the FTC from single-variable calculus. $\square$

Definition 7.4.2. *conservative vector field*

> Suppose $U \subseteq \mathbb{R}^n$ then we say $\vec{F}$ is **conservative** on U iff there exists a **potential function** f such that $\vec{F} = \nabla f$ on U. Moreover, if $\vec{F}$ is conservative on $dom(\vec{F})$ then we say $\vec{F}$ is a conservative vector field.

The beauty of a conservative vector field is we trade computation of a line-integral for evaluation at the end-points.

Example 7.4.3. *Suppose $\vec{F}(x,y,z) = \langle 2x, 2y, 3 \rangle$ for all $(x,y,z) \in \mathbb{R}^3$. Suppose C is a curve from $(0,0,0)$ to (a,b,c). Calculate $\int_C \vec{F} \cdot d\vec{r}$. Observe that*

$$f(x,y,z) = x^2 + y^2 + 3z \quad \Rightarrow \quad \vec{F} = \nabla f.$$
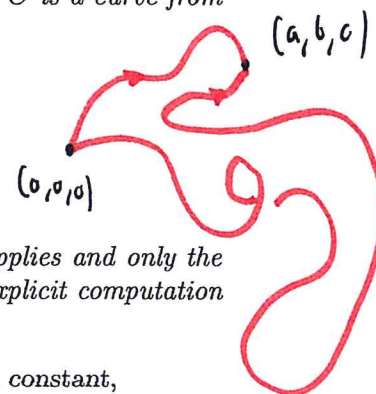
Therefore,

$$\int_C \vec{F} \cdot d\vec{r} = f(a,b,c) - f(0,0,0) = a^2 + b^2 + 3c.$$

Notice that we did not have to know where the curve C went since the FTC applies and only the endpoints of the curve are needed. In invite the reader to check this result by explicit computation along some path.

Why "conservative"? Let me address that. The key is a little identity, if m is a constant,

$$\frac{d}{dt}\left[\frac{1}{2}mv^2 \right] = \frac{d}{dt}\left[\frac{1}{2}m\vec{v} \cdot \vec{v} \right] = \frac{1}{2}m\left[\frac{d\vec{v}}{dt} \cdot \vec{v} + \vec{v} \cdot \frac{d\vec{v}}{dt} \right] = m\vec{a} \cdot \vec{v}.$$

If $\vec{F}$ is the net-force on a mass m then Newton's Second Law states $\vec{F} = m\vec{a}$ therefore, if C is a curve from $\vec{r}_1$ to $\vec{r}_2$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_1}^{t_2} \vec{F} \cdot \frac{d\vec{r}}{dt} dt = \int_{t_1}^{t_2} \left(m\vec{a} \cdot \vec{v} \right) dt = \int_{t_1}^{t_2} \frac{d}{dt}\left[\frac{1}{2}mv^2\right] dt = K(t_2) - K(t_1)$$

where $K = \frac{1}{2}mv^2$ is the kinetic energy. This result is known as the **work-energy** theorem. It does not require that $\vec{F}$ be conservative. If $\vec{F}$ is conservative then it is traditional to choose a potential energy function U such that $\vec{F} = -\nabla U$. In this case we can use the FTC for line-integrals to once more calculate the work done by the net-force,

$$\int_C \vec{F} \cdot d\vec{r} = -\int_C \nabla U \cdot d\vec{r} = -U(\vec{r}_2) + U(\vec{r}_1)$$

$$\vec{F} = -\nabla U$$
$$U = PE$$

It follows that we have, for a conservative force, $K_2 - K_1 = -U_2 + U_1$ hence $K_1 + U_1 = K_2 + U_2$. The quantity $E = U + K$ is the total mechanical energy and it is a constant of the motion when only conservative forces comprise the net-force. This is the reason I call a vector field which is a gradient field of some pontential a conservative vector field. When viewed as a net-force it provides the conservation of energy[9]. It turns out that usually we can find portions of the domain of an arbitrary vector field on which the vector field is conservative. The obstructions to the existence of a global potential are the interesting part.

Definition 7.4.4. *path-independence*

Suppose $U \subseteq \mathbb{R}^n$ then we say $\vec{F}$ is **path-independent** on U iff $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ for each pair of curves $C_1, C_2 \subset U$ beginning at P and terminating at Q .
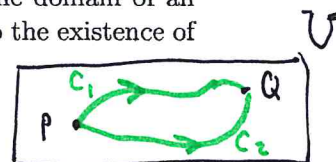
It is useful to examine a web of concepts which all serve to characterize conservative vector fields.

Proposition 7.4.5.

Suppose U is an open connected subset of $\mathbb{R}^n$ then the following are equivalent

1. $\vec{F}$ is conservative; $\vec{F} = \nabla f$ on all of U

2. $\vec{F}$ is path-independent on U

3. $\oint_C \vec{F} \cdot d\vec{r} = 0$ for all closed curves C in U

4. (add precondition $n = 3$ and U be simply connected) $\nabla \times \vec{F} = 0$ on U.

Proof: We postpone the proof of (4.) $\Rightarrow$ (1.). However, we can show that (1.) $\Rightarrow$ (4.). Suppose $\vec{F} = \nabla f$. Note that $\nabla \times \vec{F} = \nabla \times \nabla f = 0$. I included this here since we can quickly test to see if $Curl(\vec{F}) \neq 0$. When the curl is nontrivial then we can be certain the given vector field is not conservative. On the other hand, vanishing curl is only useful if it occurs over a *simply connected* domain[10]

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[9]it is worth noticing that while physically this is most interesting to three dimensions, the math allows for more
[10]a simply connected domain is a set with no holes, any loop can be smoothly shrunk to a point, it has a boundary which is a simple curve. A simple curve is a curve with no self-intersections but perhaps one in the case it is closed. A circle is simple a figure 8 is not.

(1.) $\Rightarrow$ (2.). Assume $\vec{F} = \nabla f$. Suppose C_1, C_2 are two curves which both start at P and end at Q in the set U. Apply the FTC for line-integrals in what follows:

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_1} \nabla f \cdot d\vec{r} = f(Q) - f(P).$$

Likewise, $\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_2} \nabla f \cdot d\vec{r} = f(Q) - f(P)$. Therefore (2.) holds true.

(2.) $\Rightarrow$ (1.). Assume $\vec{F}$ is path-independent. Pick some point $A \in U$ and let C be any curve in U from A to $B = (x, y, z)$. We define $f(x, y, z) = \int_C \vec{F} \cdot d\vec{r}$. This is single-valued since we assume $\vec{F}$ is path-independent. We need to show that $\nabla f = \vec{F}$. Denote $\vec{F} = \langle P, Q, R \rangle$. We begin by isolating the x-component. We need to show $\frac{\partial}{\partial x} \int_C \vec{F} \cdot d\vec{r} = P(x, y, z)$. We can write C as curve C_x from A to $B_x = (x_o, y, z)$ with $x_o < x$ pasted togther with the line-segment L_x from B_x to B. Observe that the curve C_x has no dependence on x (of the B point)

$$\frac{\partial}{\partial x} \int_C \vec{F} \cdot d\vec{r} = \frac{\partial}{\partial x} \left[\int_{C_x} \vec{F} \cdot d\vec{r} + \int_{L_x} \vec{F} \cdot d\vec{r} \right] = \frac{\partial}{\partial x} \left[\int_{L_x} \vec{F} \cdot d\vec{r} \right]$$

The line segment L_x has parametrization $\vec{r}(t) = \langle t, y, z \rangle$ for $x_o \le t \le x$. We calculate that

$$\int_{L_x} \vec{F} \cdot d\vec{r} = \int_{x_o}^{x} \vec{F}(t, y, z) \cdot \langle 1, 0, 0 \rangle dt = \int_{x_o}^{x} P(t, y, z) dt$$

Therefore,

$$\frac{\partial}{\partial x} \int_C \vec{F} \cdot d\vec{r} = \frac{\partial}{\partial x} \int_{x_o}^{x} P(t, y, z) dt = P(x, y, z).$$

We can give similar arguments to show that

$$\frac{\partial}{\partial y} \int_C \vec{F} \cdot d\vec{r} = Q \quad \& \quad \frac{\partial}{\partial z} \int_C \vec{F} \cdot d\vec{r} = R.$$

We find $\vec{F}$ is conservative.

(2.) $\Rightarrow$ (3.). Assume $\vec{F}$ is path-independent. Consider a closed curve C in U. Notice we can pick any pair of points P, Q on C and write C_1 from P to Q and C_2 from Q to P such that $C = C_1 \cup C_2$. Furthermore, note that $-C_2$ also goes from P to Q. Path independence yields
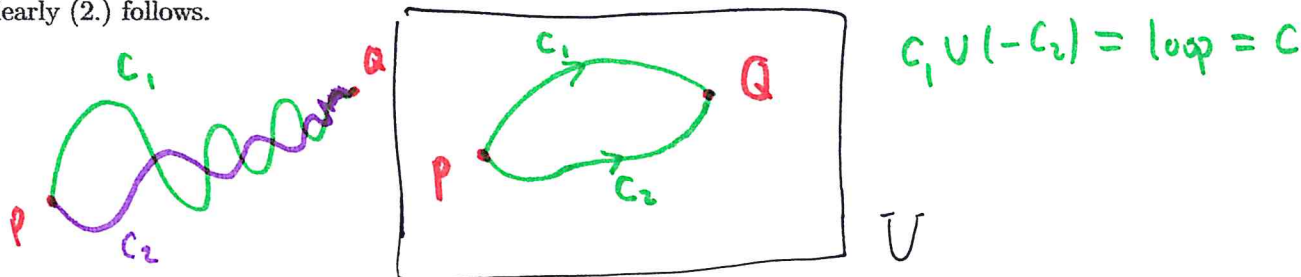
$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{-C_2} \vec{F} \cdot d\vec{r} \Rightarrow 0 = \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{-C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} = \oint_C \vec{F} \cdot d\vec{r}.$$

Conequently (3.) is true.

(3.) $\Rightarrow$ (2.). Suppose $\oint_C \vec{F} \cdot d\vec{r} = 0$ for all closed curves C in U. Suppse C_1 and C_2 start at P and end at Q. Observe that $C = C_1 \cup (-C_2)$ is a closed curve hence

$$0 = \oint_C \vec{F} \cdot d\vec{r} \Rightarrow 0 = \oint_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{-C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r}.$$

Clearly (2.) follows.

To summarize we have shown (1.) ⇔ (2.) ⇔ (3.) and (1.) ⇒ (4.). We postpone the proof that (3.) ⇒ (4.) and (4.) ⇒ (1.) . □

The point A where $f(A) = 0$ is known as the zero for the potential. You should notice that the choice for f is not unique. If we add a constant c to the potential function f then we obtain the same gradient field; $\nabla f = \nabla (f + c)$. In physics this is the freedom to set the potential energy to be zero at whichever point is convenient.

Example 7.4.6. *In electrostatics, the potential energy per unit charge is called the voltage or simply the electric potential. For finite, localized charg distributions the electric potential is defined by*

$$V(x,y,z) = -\int_{\infty}^{(x,y,z)} \vec{E} \cdot d\vec{r}$$

The electric field of a charge at the origin is given by $\vec{E} = \frac{k}{\rho^2}\hat{\rho}$. We take the line from the origin to spatial infinity[11] to calculate the potential.

$$V(\rho) = -\int_{\infty}^{\rho} \frac{k}{\rho^2} d\rho = \frac{k}{\rho}.$$

The notation $\int_{P}^{Q}$ indicates the line integral is taken over a path from P to Q. This notation is only unambiguous if we are working with a conservative vector field.

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[11]the claim implicit within such a convention is that it matters not which unbounded direction the path begins, for convenience we usually just use a line which extends to ∞