LECTURE 33: GREEN'S THEOREM

7.5 green's theorem

The fundamental theorem of calculus shows that integration and differentiation are inverse processes in a certain sense. It is natural to seek out similar theorems for functions of several variables.

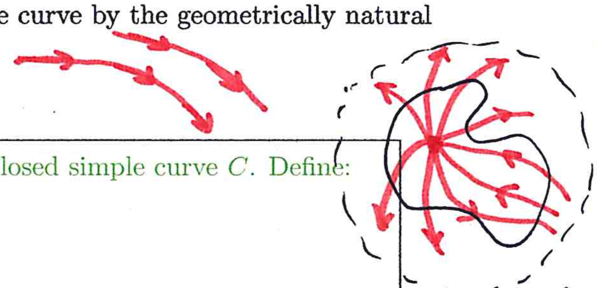
We begin our search by defining the flux through a simple closed planar curve[12]. It is just the scalar integral of the outward-facing normal component to the vector field. Then we examine how a vector field flows out of a little rectangle. This gives rises us reason to define the divergence. In some sense this little picture will derive the first form of Green's Theorem.

Being discontent with just one interpretation, we turn to analyze how the vector field circulates around a given CCW curve. We again look at a little rectangle and quantify how a given vector field twists around the square loop. This leads us to another derivation of Green's Theorem. Moreover, it gives us the reason to define the curl of a vector field.

Finally, we offer a proof which extends the toy derivations to a general Type I & II curve. Past that, properties of the line-integral extend our result to general regions in the plane. Applications to the calculation of area and the analysis of conservative vector fields are given. I conclude this section with a somewhat formal introduction to two-dimensional electrostatics, I show how Green's Theorem naturally supports Gauss' Law for the plane.

7.5.1 geometry of divergence in two dimensions

A curve is said to be **simple** if it has no self-intersections except perhaps one. For example, a circle is a simple curve whereas a figure 8 is not. Both circles and figure 8's are closed curves since they have no end points (or you could say they have the same starting and ending points). In any event, we define the number of field lines which cut through a simple curve by the geometrically natural definition below:

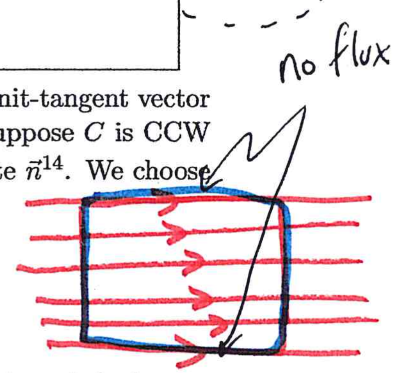


Definition 7.5.1. *flux of $\vec{F}$ through a simple curve C.*

Suppose $\vec{F}$ is is continuous on a open set containing the closed simple curve C. Define:

$$\Phi_C = \oint_C (\vec{F} \bullet \vec{n})ds$$

Where $\vec{n}$ is the outward-facing unit-normal to C.



Recall that if $\vec{r}(s) = \langle x(s), y(s)\rangle$ is the arclength parametrization of C then the unit-tangent vector of the Frenet frame was defined by $\vec{T}(s) = \frac{d\vec{r}}{ds}$.[13] For the sake of visualization suppose C is CCW oriented calculate $\vec{n}$. Since $\vec{T} \bullet \vec{n} = 0$ there are only two choices once we calculate $\vec{n}$[14]. We choose the $\vec{n}$ which points outward.

$$\vec{T} = \left\langle \frac{dx}{ds}, \frac{dy}{ds} \right\rangle \quad \Rightarrow \quad \vec{n} = \left\langle \frac{dy}{ds}, -\frac{dx}{ds} \right\rangle$$

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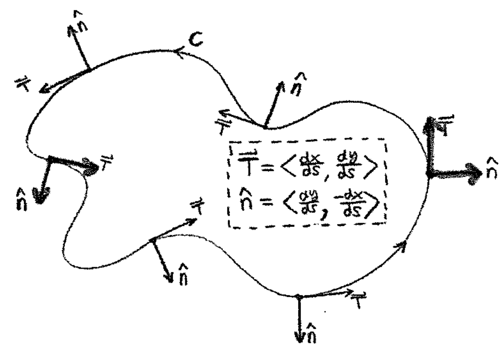
[12] these are known as Jordan curves

[13] yes the unit normal is defined by and $\vec{N}(s) = \frac{1}{T'(s)}\frac{d\vec{T}}{ds}$. However, this is not the $\vec{n}$ which we desire because $\vec{N}$ sometimes points inward. Also, direct computation brings us to second derivatives and the geometric argument above avoids that difficulty

[14] if $\hat{u} = \langle a, b\rangle$ then $\hat{v} = \langle b, -a\rangle$ or $\hat{v} = \langle -b, a\rangle$ are the only perpendicular unit-vectors to $\hat{u}$

The picture below helps you see how the outward normal formula works:



$$\vec{T} = \langle \frac{dx}{ds}, \frac{dy}{ds} \rangle$$

$$\hat{n} = \langle \frac{dy}{ds}, -\frac{dx}{ds} \rangle$$

$$\vec{T} \cdot \hat{n} = 0$$

Let's calculate the flux given this identity. Consider a vector field $\vec{F} = \langle P, Q \rangle$ and once more the Jordan curve C with outward normal $\vec{n}$, suppose length of C is L,

$$\begin{aligned}
\Phi_C &= \oint_C (\vec{F} \bullet \vec{n})\, ds \\
&= \int_0^L \langle P, Q \rangle \bullet \langle \frac{dy}{ds}, -\frac{dx}{ds} \rangle ds \\
&= \int_0^L \left(P\frac{dy}{ds} - Q\frac{dx}{ds} \right) ds \\
&= \int_0^L \langle -Q, P \rangle \bullet \langle \frac{dx}{ds}, \frac{dy}{ds} \rangle ds \\
&= \oint_C Pdy - Qdx. \quad \longleftarrow \text{calculates flux of } \langle P, Q \rangle
\end{aligned}$$

This formula is very nice. It equally well applies to closed simple curves which are only mostly smooth. If we have a few corners on C then we can still calculate the flux by calculating flux through each smooth arc and adding together to find the net-flux. To summarize:

Proposition 7.5.2.

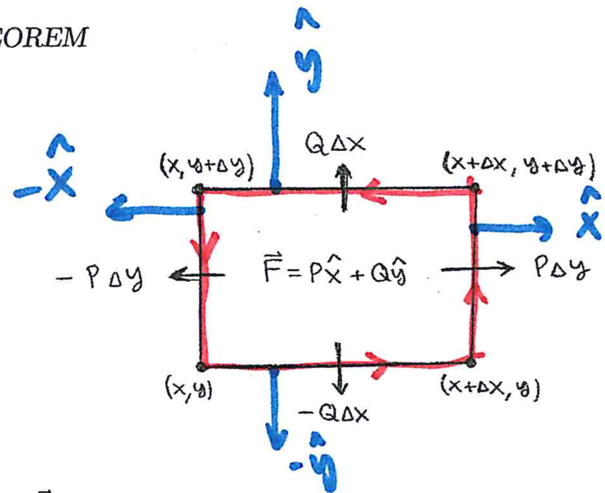
> Suppose C is a piecewise-smooth, simple, closed CCW oriented curve. If $\vec{F}$ is continuous on an open set containing C then the flux through C is given by
>
> $$\Phi_C = \oint_C \vec{F} \bullet \vec{n} = \oint_C Pdy - Qdx.$$

If you were to consider the CW-oriented curve $-C$ then the outward-normal is given by $\vec{n} = \langle -\frac{dy}{ds}, \frac{dx}{ds} \rangle$ and the formula for flux is

$$\Phi_C = \oint_{-C} \vec{F} \bullet \vec{n} = \oint_{-C} Qdx - Pdy.$$

This formula is in some sense left-handed, hence evil, so we whilst not use it hence forth.

Now we turn to the task of approximating the flux by direct computation. Consider a little rectangle R with corners at $(x, y), (x + \triangle x, y), (x + \triangle x, y + \triangle y), (x, y + \triangle y)$.



$$\vec{F} = \langle P, Q \rangle = P\hat{x} + Q\hat{y}$$

To calculate the flux of $\vec{F} = \langle P, Q \rangle$ through the rectangle we simply find the flux through each side and add it up.

1. **Top:** $(\vec{F} \bullet \hat{y})\Delta x = Q(x, y + \Delta y)\Delta x$
2. **Base:** $(\vec{F} \bullet [-\hat{y}])\Delta x = -Q(x, y)\Delta x$
3. **Left:** $(\vec{F} \bullet [-\hat{x}])\Delta y = -P(x, y)\Delta y$
4. **Right:** $(\vec{F} \bullet \hat{x})\Delta y = P(x + \Delta x, y)\Delta y$

The net-flux through R is thus,

$$\Phi_R = \Big(Q(x, y + \Delta y) - Q(x, y)\Big)\Delta x + \Big(P(x + \Delta x, y) - P(x, y)\Big)\Delta y$$

Observe that

$$\frac{\Phi_R}{\Delta x \Delta y} = \frac{Q(x, y + \Delta y) - Q(x, y)}{\Delta y} + \frac{P(x + \Delta x, y) - P(x, y)}{\Delta x}$$

In this limit $\Delta x \to 0$ and $\Delta y \to 0$ the expressions above give partial derivatives and we find that:

$$\frac{d\Phi_R}{dA} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \nabla \bullet \langle P, Q \rangle \qquad \int d\Phi_R = \iint (P_x + Q_y)\, dA$$

This suggests we define the flux density as $\nabla \bullet \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$. If we integrate this density over a finite region then we will find the net flux through the region (jump!). In any event, we at least have good reason to suspect that

$$\Phi_{\partial R} = \oint_{\partial R} \vec{F} \bullet \vec{n} = \iint_R \nabla \bullet \vec{F}\, dA \quad \Rightarrow \quad \boxed{\oint_{\partial R} P dy - Q dx = \iint_R \left[\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right] dA}$$

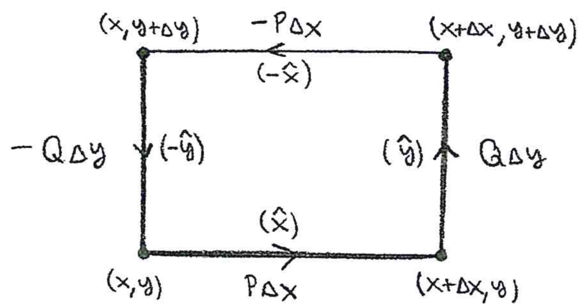
This is for obvious reasons called the *divergence form of Green's Theorem.* We prove this later in this section. As I mentioned in lecture I found these thoughts in Thomas' Calculus text, however, I suspect we'll find them in many good calculus texts at this time.

Compare to

$$\oint_{\partial R} P dx + Q dy = \iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA$$

7.5.2 geometry of curl in two dimensions

Recall that $\oint_C \vec{F} \cdot d\vec{r}$ calculates the work done by $\vec{F}$ around the loop C. This line-integral is also called the **circulation**. Why? If we think of $\vec{F}$ as the velocity field of some liquid then a positive circulation around a CCW loop suggests that a little paddle wheel placed at the center of the loop will spin in the CCW direction. The greater the circulation the faster it spins. Let us duplicate the little rectangle calculation of the previous section to see what meaning, if any, the circulation per area has: Once more, consider a little rectangle R with corners at $(x,y), (x+\Delta x, y), (x+\Delta x, y+\Delta y), (x, y+\Delta y)$.



To calculate the flow[15] of $\vec{F} = \langle P, Q\rangle$ through the rectangle we simply find the flow through each side and add it up.

1. **Top:** $(\vec{F} \cdot [-\hat{x}])\Delta x = -P(x, y+\Delta y)\Delta x$.
2. **Base:** $(\vec{F} \cdot \hat{x})\Delta x = P(x,y)\Delta x$
3. **Left:** $(\vec{F} \cdot [-\hat{y}])\Delta y = -Q(x,y)\Delta y$
4. **Right:** $(\vec{F} \cdot \hat{y})\Delta y = Q(x+\Delta x, y)\Delta y$

The net-circulation around R is thus,

$$W_R = \Big(Q(x+\Delta x, y) - Q(x,y)\Big)\Delta y - \Big(P(x, y+\Delta y) - P(x,y)\Big)\Delta x$$

Observe that

$$\frac{W_R}{\Delta x \Delta y} = \frac{Q(x+\Delta x, y) - Q(x,y)}{\Delta x} - \frac{P(x,y+\Delta y) - P(x,y)}{\Delta y}$$

In this limit $\Delta x \to 0$ and $\Delta y \to 0$ the expressions above give partial derivatives and we find that:

$$\frac{dW_R}{dA} \quad \frac{d\Phi_R}{dA} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \left(\nabla \times \langle P, Q, 0\rangle\right)_3 = \left(\langle 0, 0, Q_x - P_y\rangle\right)_3 = Q_x - P_y.$$

In the case that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ we say the velocity field is *irrotational* since it does not have the tendency to generate rotation at the point in question. The **curl** of a vector field measures

[15] the flow around a closed loop is called circulation, but flow is the term for a curve which is not closed

how a vector field in $\mathbb{R}^3$ rotates about planes with normals $\widehat{x}, \widehat{y}, \widehat{z}$. In particular, we defined $Curl(\vec{F}) = \nabla \times \vec{F}$. If $\vec{F}(x,y,z) = \langle P(x,y), Q(x,y), 0\rangle$ then

$$Curl(\vec{F}) = \left\langle\, 0,\ 0,\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

and we just derived that nonzero $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ will spin a little paddle wheel with axis $\widehat{z}$. If $\vec{F} \bullet \widehat{z} \neq 0$ and/or P, Q had nontrivial z-dependence then we would also find nontrivial components of the curl in the $\widehat{x}$ or $\widehat{y}$ directions. If $Curl(\vec{F}) \bullet \widehat{x}$ or $Curl(\vec{F}) \bullet \widehat{y}$ were nonzero at a point then that suggests the vector field will spin a little paddle wheel with axis $\widehat{x}$ or $\widehat{y}$. That is clear from simply generalizing this calculation by replacing x, y with y, z or x, z. Another form of Green's Theorem follows from the curl: since $dW = (\nabla \times \vec{F}) \bullet \widehat{z}\, dA$ we suspect that

$$W_R = \oint_{\partial R} \vec{F} \bullet d\vec{r} = \iint_R (\nabla \times \vec{F}) \bullet \widehat{z}\, dA \quad \Rightarrow \quad \boxed{\oint_{\partial R} P dx + Q dy = \iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA}$$

This is the more common form found in calculus texts. Many texts simply state this formula and offer part of the proof given in the next section. Our goal here was to understand why we would expect such a theorem and as an added benefit we have hopefully arrived at a deeper understanding of the differential vector calculus of curl and divergence.

7.5.3 proof of the theorem

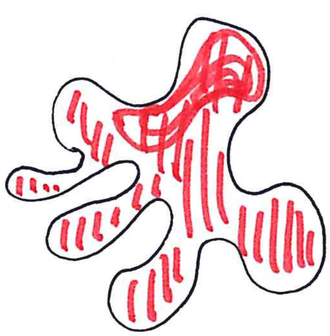
It it is a simple exercise to show that the divergence form of Green's theorem follows from the curl-form we state below.

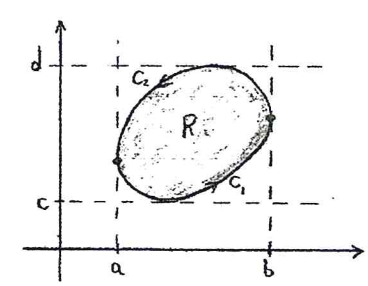
Theorem 7.5.3. *Green's Theorem for simply connected region:*

Suppose ∂R is a piecewise-smooth, simple, closed CCW oriented curve which bounds the simply connected region $R \subset \mathbb{R}^2$ and suppose $\vec{F}$ is differentiable on an open set containing R then

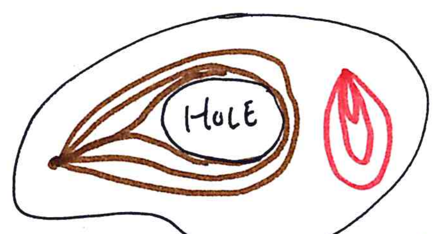
$$\oint_{\partial R} P dx + Q dy = \iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA$$

Proof: we begin by observing that any simply connected region can be subdivided into more basic simply connected regions which are simultaneously Type I and Type II subsets of the plane. Sometimes, it takes a sequence of these basic regions to capture the overall region R and we will return to this point once the theorem is settled for the basic case.





$\partial R = C_1 \cup C_2$



not simply connected

Proof of Green's Theorem for regions which are both type I and II. We assume that there exist constants $a, b, c, d \in \mathbb{R}$ and functions f_1, f_2, g_1, g_2 which are differentiable and describe R as follows:

$$R = \underbrace{\{(x,y) \mid a \leq x \leq b,\ f_1(x) \leq y \leq f_2(x)\}}_{\text{type I}} = \underbrace{\{(x,y) \mid c \leq y \leq d,\ g_1(y) \leq x \leq g_2(y)\}}_{\text{type II}}$$

Note the boundary $\partial R = C_1 \cup C_2$ can be parametrized in the type I set-up as:

$$C_1 : \vec{r}_1(x) = \langle x, f_1(x)\rangle, \qquad -C_2 : \vec{r}_{-2}(x) = \langle x, f_2(x)\rangle$$

for $a \leq x \leq b$ (it is easier to think about parametrizing $-C_2$ so I choose to do such). Proof of the theorem can be split into proving two results:

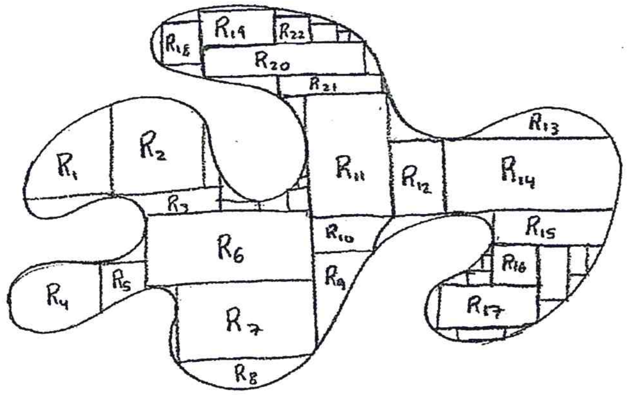
$$\textbf{(I.)} \ \oint_{\partial R} P dx = -\iint_R \frac{\partial P}{\partial y} dA \qquad \& \qquad \textbf{(II.)} \ \oint_{\partial R} Q dy = \iint_R \frac{\partial Q}{\partial x} dA$$

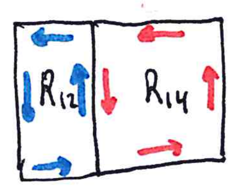
I prove $I.$ in these notes and I leave II. as a homework for the reader. Consider,

$$\begin{aligned}
\iint_R \frac{\partial P}{\partial y} dA &= \int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial P}{\partial y} dy\, dx \\
&= \int_a^b [\ P(x, f_2(x)) - P(x, f_1(x))\ ] dx \\
&= \int_a^b P(x, f_2(x)) dx - \int_a^b P(x, f_1(x)) dx \\
&= \int_{-C_2} P dx - \int_{C_1} P dx \\
&= -\left(\int_{C_2} P dx + \int_{C_1} P dx\right) \qquad C = C_1 \cup C_2 \\
&= -\int_C P dx
\end{aligned}$$

Hence $\iint_R -\frac{\partial P}{\partial y} dA = \oint_C P dx$. You will show in homework that $\oint_{\partial R} Q dy = \iint_R \frac{\partial Q}{\partial x} dA$ and Green's Theorem for regions which are both type I and II follows. ∇

If the set R is a simply connected subset of the plane then it has no holes and generically the picture is something like what follows. We intend that $R = \sum_k R_k$





Applying Green's theorem to each sub-region gives us the following result.

$$\sum_k \oint_{\partial R_k} P dx + Q dy = \sum_k \iint_{R_k} \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA$$

It is geometrically natural to suppose the rhs simply gives us the total double integral over R,

$$\sum_k \iint_{R_k} \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA = \iint_{R} \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA.$$

I invite the reader to consider the diagram above to see that all the interior *cross-cuts* cancel and only the net-boundary contributes to the line integral over ∂R. Hence,

$$\sum_k \oint_{\partial R_k} P dx + Q dy = \oint_{\partial R} P dx + Q dy.$$

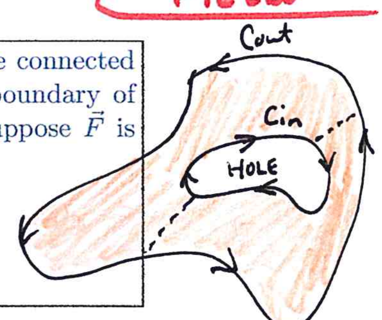
Green's Theorem follows. It should be cautioned that the summations above need not be finite. We neglect some analytical details in this argument. However, I hope the reader sees the big idea here. You can find full details in some advanced calculus texts. □

LECTURE 34: DEFORMATION THEOREM & CONSERVATIVE VECTOR FIELD

Theorem 7.5.4. *Green's Theorem for an annulus:*

Suppose ∂R is a pair of simple, closed ~~CCW~~ oriented curves which bounds the connected region $R \subset \mathbb{R}^2$ where $\partial R = C_{in} \cup C_{out}$ and C_{in} is the CW-oriented inner-boundary of R whereas C_{out} is the CCW oriented outer-boundary of R. Furthermore, suppose $\vec{F}$ is differentiable on an open set containing R then

$$\oint_{\partial R} P dx + Q dy = \iint_{R} \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA$$



Proof: See the picture below we can break the annulus into two simply connected regions then apply Green's Theorem for simply connected regions to each piece.

$\partial R = C_{out} \cup C_{in}$