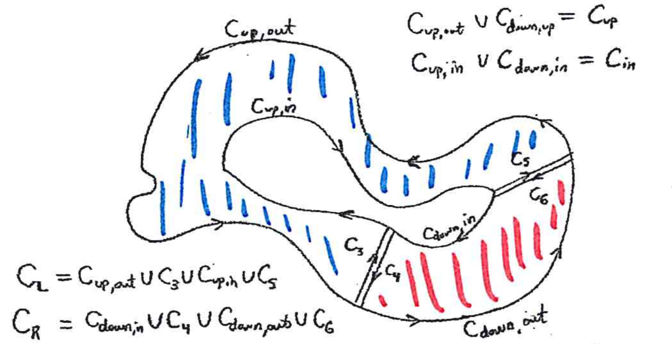


$$\int_{C_3} \vec{F}\cdot d\vec{r} = -\int_{C_4} \vec{F}\cdot d\vec{r} \quad \text{and} \quad \int_{C_5} \vec{F}\cdot d\vec{r} = -\int_{C_6} \vec{F}\cdot d\vec{r}$$

Observe that the cross-cuts cancel (in the diagram above the cancelling pairs are $3, 4$ and $5, 6$):

$$\begin{aligned}
\int_{C_L} \vec{F}\cdot d\vec{r} + \int_{C_R} \vec{F}\cdot d\vec{r} &= \int_{C_{up,out}} \vec{F}\cdot d\vec{r} + \int_{C_3} \vec{F}\cdot d\vec{r} + \int_{C_{up,in}} \vec{F}\cdot d\vec{r} + \int_{C_5} \vec{F}\cdot d\vec{r} \\
&\quad + \int_{C_{down,in}} \vec{F}\cdot d\vec{r} + \int_{C_4} \vec{F}\cdot d\vec{r} + \int_{C_{down,out}} \vec{F}\cdot d\vec{r} + \int_{C_6} \vec{F}\cdot d\vec{r} \\
&= \int_{C_{up,out}} \vec{F}\cdot d\vec{r} + \int_{C_3} \vec{F}\cdot d\vec{r} + \int_{C_{up,in}} \vec{F}\cdot d\vec{r} + \int_{C_5} \vec{F}\cdot d\vec{r} \\
&\quad + \int_{C_{down,in}} \vec{F}\cdot d\vec{r} - \int_{C_3} \vec{F}\cdot d\vec{r} + \int_{C_{down,out}} \vec{F}\cdot d\vec{r} - \int_{C_5} \vec{F}\cdot d\vec{r} \\
&= \int_{C_{up,out}} \vec{F}\cdot d\vec{r} + \int_{C_{up,in}} \vec{F}\cdot d\vec{r} + \int_{C_{down,in}} \vec{F}\cdot d\vec{r} + \int_{C_{down,out}} \vec{F}\cdot d\vec{r} \\
&= \int_{C_{up}} \vec{F}\cdot d\vec{r} + \int_{C_{down}} \vec{F}\cdot d\vec{r}
\end{aligned}$$

Apply Green's theorem to the regions bounded by C_L and C_R and the theorem follows. □

$\partial R = C_{out} \cup C_{in}$

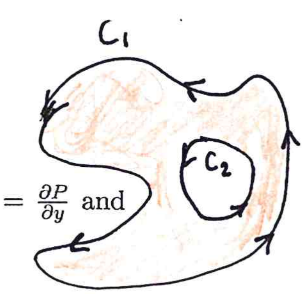
Notice we can recast this theorem as follows:

C_{in} CW $\Rightarrow -C_{in}$ CCW.

$$\oint_{C_{out}} Pdx+Qdy + \oint_{C_{in}} Pdx+Qdy = \oint_{C_{out}} Pdx + Qdy - \oint_{-C_{in}} Pdx + Qdy = \iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA.$$

Or, better yet, if C_1, C_2 are two CCW oriented curves which bound R

$$\oint_{C_1} Pdx + Qdy - \oint_{C_2} Pdx + Qdy = \iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA.$$



Suppose the vector field $\vec{F} = \langle P, Q \rangle$ passes our *Clairaut Test* on R then we have $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ and consequently:

$$\boxed{\oint_{C_1} Pdx + Qdy = \oint_{C_2} Pdx + Qdy.}$$

I often refer to this result as the **deformation theorem** for irrotational vector fields in the plane.

Theorem 7.5.5. *Deformation Theorem for irrotational vector field on the plane:*

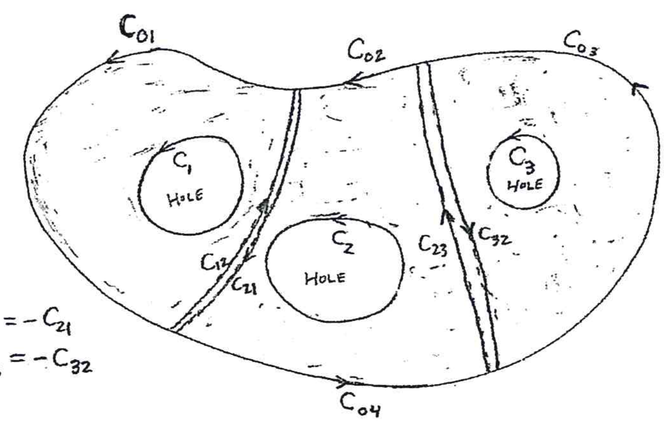
> Suppose C_1, C_2 are CCW oriented closed simple curves which bound $R \subset \mathbb{R}^2$ and suppose $\vec{F} = \langle P, Q \rangle$ is differentiable on a open set containing R then
>
> $$\oint_{C_1} Pdx + Qdy = \oint_{C_2} Pdx + Qdy.$$

In my view, points where $\nabla \times \vec{F} \neq 0$ are troublesome. This theorem says the line integral is unchanged if we do not enclose any new troubling points as we deform C_1 to C_2. On the flip-side of this, if the integral around some loop is nonzero for a given vector field that must mean that something interesting happens to the curl of the vector field on the interior of the loop.

Theorem 7.5.6. *Green's Theorem for a region with lots of holes.*

> Suppose R is a connected subset of $\mathbb{R}^2$ which has boundary ∂R. We orient this boundary curve such that the outer boundary has CCW orientation whereas all the inner-boundaries have CW orientation. Furthermore, suppose $\vec{F}$ is differentiable on an open set containing R then
>
> $$\oint_{\partial R} Pdx + Qdy = \iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA$$

Proof: follows from the picture below and a little thinking.



$C_{12} = -C_{21}$

$C_{23} = -C_{32}$

$$C_{out} = C_{04} \cup C_{03} \cup C_{02} \cup C_{01}$$

$$C_{1D} = C_{01} \cup C_{12}$$

$$C_{2D} = C_{02} \cup C_{21} \cup C_{04} \cup C_{23}$$

$$C_{3D} = C_{03} \cup C_{32}$$

This is more interesting if we state it in terms of the outer loop and CCW oriented inner loops. Denote C_{out} for the outside loop of ∂R and C_k for $k = 1, 2, \ldots, N$ for the inner CCW oriented loops. Since $\partial R = C_{out} \cup -C_1 \cup \cdots \cup -C_N$ it follows

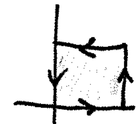
$$\oint_{C_{out}} Pdx + Qdy - \sum_{k=1}^{N} \oint_{C_k} Pdx + Qdy = \iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA$$

If we have $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ throughout R we find the following beautiful generalization of the deformation theorem:

$$\oint_{C_{out}} Pdx + Qdy = \sum_{k=1}^{N} \oint_{C_k} Pdx + Qdy.$$

In words, the net circulation around C is simply a sum of the circulations around each singularity contained within C. A **singularity** is a point at which the field $\langle P, Q\rangle$ obtains a nontrivial circulation around any small loop containing the point.

$$\int_{\partial R} Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$



7.5.4 examples

Example 7.5.7. *Use Green's theorem to calculate $\oint_C x^3dx + yxdy$ where C is the CCW boundary of the oriented rectangle R: $[0,1]\times[0,1]$. Identify that $P = x^3$ and $Q = xy$. Applying Green's Theorem,*

$$\oint_C x^3dx + yxdy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_0^1 \int_0^1 y\,dx\,dy = \frac{1}{2}.$$

One important application of Green's theorem involves the calculation of areas. Note that if we choose $\vec{F} = \langle P, Q\rangle$ such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ then the double integral in Green's theorem represents the area of R. In particular, it is common to use

$$\vec{F} = \langle 0, x\rangle, \qquad \text{or} \qquad \vec{F} = \langle -y, 0\rangle, \qquad \text{or} \qquad \vec{F} = \langle -y/2, x/2\rangle$$

in Green's theorem to obtain the identities:

$$\boxed{A_R = \oint_{\partial R} xdy = -\oint_{\partial R} ydx = \frac{1}{2}\oint_{\partial R} xdy - ydx}$$

Example 7.5.8. *Find the area of the ellipse bounded by $x^2/a^2 + y^2/b^2 = 1$. Observe that the ellipse ∂R is parametrized by $x = a\cos(t)$ and $y = b\sin(t)$ hence $dx = -a\sin(t)dt$ and $dy = b\cos(t)dt$ hence*

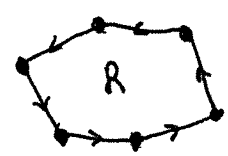
$$A_R = \frac{1}{2}\oint_{\partial R} xdy - ydx = \frac{1}{2}\int_0^{2\pi} 1a\cos(t)b\cos(t)dt - b\sin(t)(-a\sin(t)dt) = \frac{1}{2}\int_0^{2\pi} ab\,dt = \pi ab.$$

When $a = b = R$ we obtain the famous πR^2.

Example 7.5.9. *You can show (perhaps you will in a homework) that for the line-segment L from (x_1, y_1) to (x_2, y_2) we have the following excellent identity:*

$\partial R = \bigcup$ line segs

$$\frac{1}{2}\int_L xdy - ydx = \frac{1}{2}(x_1y_2 - x_2y_1).$$



Consider that if P is a polygon with vertices $(x_1, y_1), (x_2, y_2), \dots, (x_N, y_N)$ with sides L_{12}, L_{20} [3] $\dots, L_{N-1,N}$ then the area of P is given by the line-integral of $\vec{F} = \langle -y/2, x/2\rangle$ thanks to Green's Theorem:

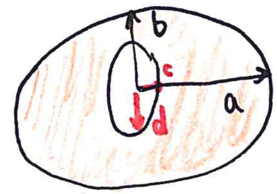
typo

$$\begin{aligned} A_P = \iint_P dA &= \frac{1}{2}\int_{\partial P} xdy - ydx \\ &= \frac{1}{2}\int_{L_{12}} xdy - ydx + \frac{1}{2}\int_{L_{23}} xdy - ydx + \dots + \frac{1}{2}\int_{L_{N-1,N}} xdy - ydx \\ &= \frac{1}{2}\big[x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + \dots + x_{N-1}y_N - x_Ny_{N-1}\big] \end{aligned}$$

You can calculate the area of polygon with vertices $(0,0), (-1,1), (0,2), (1,3), (2,1)$ is $9/2$ by applying the formula above. You could just as well calculate the area of a polygon with 100 vertices. The example below is a twist on the ellipse example already given. This time we study an annulus with elliptical edges.

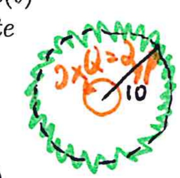
Example 7.5.10. *Find the area bounded by ellipses $x^2/a^2+y^2/b^2=1$ and $x^2/c^2+y^2/d^2=1$ given that $0<c<a$ and $0<d<b$ to insure that the ellipse $x^2/a^2+y^2/b^2=1$ is exterior to the ellipse $x^2/c^2+y^2/d^2=1$. Observe that the ellpitical annulus has boundary $\partial R = C_{in} \cup C_{out}$ where C_{out} is CCW parametrized by $x = a\cos(t)$ and $y = b\sin(t)$ and C_{in} is CW oriented with parametrization $x = c\cos(t)$ and $y = -d\sin(t)$ it follows that:*

$$\begin{aligned} A_R &= \frac{1}{2}\oint_{\partial R} xdy - ydx \\ &= \frac{1}{2}\oint_{C_{out}} xdy - ydx + \frac{1}{2}\oint_{C_{in}} xdy - ydx \\ &= \frac{1}{2}\int_0^{2\pi} ab\,dt - \frac{1}{2}\int_0^{2\pi} cd\,dt \\ &= \pi ab - \pi cd. \end{aligned}$$



Notice the CW orientation is what caused us to subtract the inner area which is missing from the annulus.

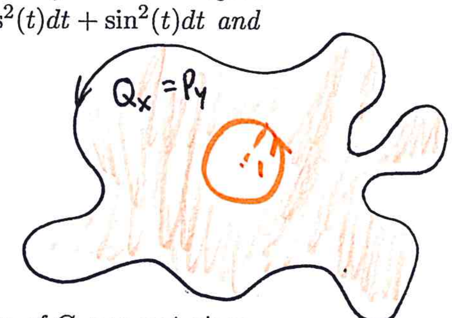
Example 7.5.11. *Consider the CCW oriented curve C with parametrization $x = (10+\sin(30t))\cos(t)$ and $y = (10+\sin(30t))\sin(t)$ for $0 \leq t \leq 2\pi$. This is a wiggly circle with mean radius 10. Calculate*



$$\int_C \frac{xdy - ydx}{x^2+y^2}.$$

Let $P = -y/(x^2+y^2)$ and $Q = x/(x^2+y^2)$ you can show that $\partial_x Q - \partial_y P = 0$ for $(x,y) \neq (0,0)$. It follows that we can deform the given problem to the simpler task of calculating the line-integral around the unit circle S_1: $x = \cos(t)$ and $y = \sin(t)$ hence $xdy - ydx = \cos^2(t)dt + \sin^2(t)dt$ and $x^2+y^2 = 1$ on S_1, calculate,

$$\begin{aligned} \int_C \frac{xdy - ydx}{x^2+y^2} &= \int_{S_1} \frac{xdy - ydx}{x^2+y^2} \\ &= \int_0^{2\pi} \frac{dt}{1} \\ &= 2\pi. \end{aligned}$$



Notice that we could still make this calculation if the specific parametrization of C was not given. Also, generally, when faced with this sort of problem we should try to pick a deformation which makes the integration easier. It was wise to deform to a circle here since the denominator was greatly simplified.

7.5.5 conservative vector fields and green's theorem

Recall that Proposition 7.4.5 gave us a list of ways of thinking about a conservative vector field:

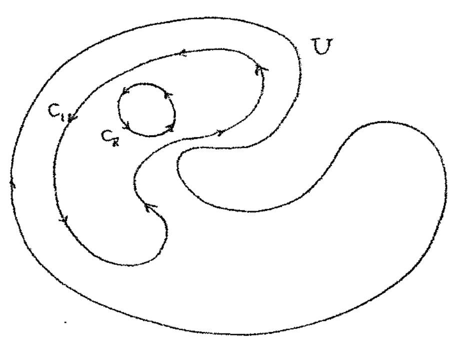
Suppose U is an open connected subset of $\mathbb{R}^n$ then the following are equivalent

1. $\vec{F}$ is conservative; $\vec{F} = \nabla f$ on all of U
2. $\vec{F}$ is path-independent on U
3. $\oint_C \vec{F} \cdot d\vec{r} = 0$ for all closed curves C in U
4. (add precondition U be simply connected) $\nabla \times \vec{F} = 0$ on U.

Proof: We finish the proof of by addressing why (4.) $\Rightarrow$ (1.) in light of Green's Theorem. Suppose U is simply connected and $\nabla \times \vec{F} = 0$ on U. Let C_1 be a closed loop in U and let C_R be another loop of radius R inside C_1. Since $\nabla \times \vec{F} = 0$ it follows we can apply the deformation Theorem 7.5.5 on the annulus between C_1 and C_R to obtain $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_R} \vec{F} \cdot d\vec{r}$. Now, as U is simply connected we can smoothly deform C_R to a point C_0. You can show that

$$\lim_{R \to 0} \int_{C_R} \vec{F} \cdot d\vec{r} = 0$$

since C_R becomes a point in this limit. Not convinced? Consider that the integral $\int_{C_R} \vec{F} \cdot d\vec{r}$ is at most the product of $max\{||\vec{F}(\vec{r})|| \mid \vec{r} \in C_R\}$ and the total arclength of C_R. However, the magnitude is bounded as $\vec{F}$ has continuous component functions and the arclength of C_R clearly goes to zero as $R \to 0$. Perhaps the picture below helps communicate the idea of the proof:



We find (4.) $\Rightarrow$ (3.) hence, by our earlier work, (3.) $\Rightarrow$ (2.) $\Rightarrow$ (1.). $\square$

Now, in terms of logical minimalism, to prove that $1, 2, 3, 4$ are equivalent we could just prove the string of implications (1.) $\Rightarrow$ (2.) $\Rightarrow$ (3.) $\Rightarrow$ (4.) $\Rightarrow$ (1.) then any of the reverse implications are easily found by logic. For example, (3.) $\Rightarrow$ (2.) would follow from (3.) $\Rightarrow$ (4.) $\Rightarrow$ (1.) $\Rightarrow$ (2.). That said, I tried to give all directions in the proof to better illustrate how the different views of the conservative vector field are connected.

7.5.6 two-dimensional electrostatics

The fundamental equation of electrostatics is known as *Gauss' Law*. In three dimensions it simply states that the flux through a closed surface is proportional to the charge which is enclosed. We have yet to define flux through a surface, but we do have a careful definition of flux through a simple closed curve. If there was a Gauss' Law in two dimensions then it ought to state that

$$\Phi_E = Q_{enc}$$

In particular, if we denote $\sigma = dQ/dA$ and have in mind the region R with boundary ∂R,

$$\oint_{\partial R} (\vec{E} \bullet \widehat{n}) ds = \iint_R \sigma dA$$

Suppose we have an isolated charge Q at the origin and we apply Gauss law to a circle of radius r centered at the origin then we can argue by symmetry the electric field must be entirely radial in direction and have a magnitude which depends only on r. It follows that:

$$\oint_{\partial R} (\vec{E} \bullet \widehat{n}) ds = \iint_R \sigma dA \quad \Rightarrow \quad (2\pi r)E = Q$$

Hence, the **coulomb field** in two dimensions is as follows:

$$\boxed{\vec{E}(r,\theta) = \frac{Q}{2\pi r}\widehat{r}}$$

Let us calculate the flux of the Coulomb field through a circle C of radius R:

$$\begin{aligned}
\oint_C (\vec{E} \bullet \widehat{n}) ds &= \int_C \left(\frac{Q}{2\pi r}\widehat{r} \bullet \widehat{r} \right) ds \\
&= \int_C \frac{Q}{2\pi R} ds \\
&= \frac{Q}{2\pi R} \int_C ds \\
&= \frac{Q}{2\pi R}(2\pi R) \\
&= Q.
\end{aligned}$$

The circle is complete. In other words, the Coulomb field derived from Gauss' Law does in fact satisfy Gauss Law in the plane. This is good news. Let's examine the divergence of this field. It appears to point away from the origin and as you get very close to the origin the magnitude of E is unbounded. It will be convenient to reframe this formula for the Coulomb field by:

$$\vec{E}(x,y) = \frac{Q}{2\pi(x^2+y^2)}\langle x, y\rangle.$$

Note:

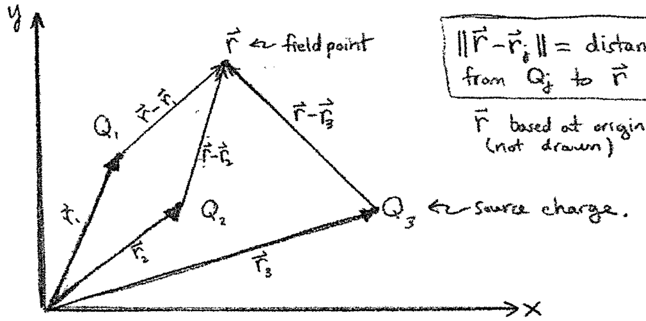
$$\begin{aligned}
\nabla \bullet \vec{E} &= \frac{\partial}{\partial x}\left[\frac{xQ}{2\pi(x^2+y^2)}\right] + \frac{\partial}{\partial y}\left[\frac{yQ}{2\pi(x^2+y^2)}\right] \\
&= \frac{Q}{2\pi}\left[\frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right] = 0.
\end{aligned}$$

If we were to carelessly apply the divergence form of Green's theorem this could be quite unsettling: consider,

$$\oint_{\partial R} \vec{E} \bullet \vec{n} = \iint_R \nabla \bullet \vec{E}\, dA \quad \Rightarrow \quad Q = \iint_R (0) dA = 0.$$

But, Q need not be zero hence there is some contradiction? Why is there no contradiction? Can you resolve this paradox?

Moving on, suppose we have N charges placed at source points $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N$ then we can find the total electric field by the principle of superposition.



We simply take the vector sum of all the coulomb fields. In particular,

$$\vec{E}(\vec{r}) = \sum_{j=1}^{N} \vec{E}_j = \sum_{j=1}^{n} \frac{Q_j}{2\pi} \frac{\vec{r} - \vec{r}_j}{||\vec{r} - \vec{r}_j||^2}$$

What is the flux through a circle which encloses just the k-th one of these charges? Suppose C_R is a circle of radius R centered at $\vec{r}_k$. We can calculate that

$$\oint_{C_R} (\vec{E}_k \bullet \hat{n}) ds = Q_k$$

whereas, since $\vec{E}_j$ is differentiable inside all of C_R for $j \neq k$ and $\nabla \bullet \vec{E}_j = 0$ we can apply the divergence form of Green's theorem to deduce that

$$\oint_{C_R} (\vec{E}_j \bullet \hat{n}) ds = 0.$$

Therefore, summing these results together we derive for $\vec{E} = \vec{E}_1 + \cdots + \vec{E}_k + \cdots + \vec{E}_N$ that

$$\oint_{C_R} (\vec{E} \bullet \hat{n}) ds = Q_k$$

Notice there was nothing particularly special about Q_k so we have derived this result for each charge in the distribution. If we take a circle around a charge which contains just one charge then Gauss' Law applies and the flux is simply the charge enclosed. Denote $C_1, C_2, \dots, C_N$ as little circles which each enclose a single charge. In particular, $C_1, C_2, \dots, C_N$ enclose the charges $Q_1, Q_2, \dots, Q_N$ respective. We have

$$Q_1 = \oint_{C_1} (\vec{E} \bullet \hat{n}) ds, \quad Q_2 = \oint_{C_2} (\vec{E} \bullet \hat{n}) ds, \quad \dots, \quad Q_N = \oint_{C_N} (\vec{E} \bullet \hat{n}) ds$$

Now suppose we have a curve C which encloses all N of the charges. The electric field is differentiable and has vanishing divergence at all points except the location of the charges. In fact, the coulomb field passes Clairaut's test everywhere. It just has the isolated singularity where the charge is found. We can apply the general form of the deformation theorem to arrive at Gauss' Law for the distribution of N-charges:

$$\oint_C (\vec{E} \bullet \widehat{n})ds = \oint_{C_1} (\vec{E} \bullet \widehat{n})ds + \oint_{C_2} (\vec{E} \bullet \widehat{n})ds + \cdots + \oint_{C_N} (\vec{E} \bullet \widehat{n})ds = Q_1 + Q_2 + \cdots + Q_N$$

You can calculate the divergence is zero everywhere except at the location of the source charges. Moral of story: even one point thrown out of a domain can have dramatic and global consequences for the behaviour of a vector field. In physics literature you might find the formula to describe what we found by a *dirac-delta function* these distributions capture certain infinities and let you work with them. For example: for the basic coulomb field with a single point charge at the origin $\vec{E}(r,\theta) = \frac{Q}{2\pi r}\widehat{r}$ this derived from a charge density function σ which is zero everywhere except at the origin. Somehow $\iint_R \sigma dA = Q$ for any region R which contains $(0,0)$. Define $\sigma(\vec{r}) = Q\delta(\vec{r})$. Where we define: for any function f which is continuous near 0 and any region which contains the origin

$$\int_R f(\vec{r})\delta(\vec{r})dA = f(0)$$

and if R does not contain $(0,0)$ then $\iint_R f(\vec{r})\delta(\vec{r})dA = 0$. The dirac delta function turns integration into evaluation. The dirac delta function is not technically a function, in some sense it is zero at all points and infinite at the origin. However, we insist it is manageably infinity in the way just described. Notice that it does at least capture the right idea for density of a point charge: suppose R contains $(0,0)$,

$$\iint_R \sigma dA = \iint_R Q\delta(\vec{r})dA = Q.$$

On the other hand, we can better understand the divegence calculation by the following calculations[16]:

$$\nabla \bullet \frac{\vec{r}}{r^2} = 2\pi\delta(\vec{r}).$$

Consequentlly, if $\vec{E} = \frac{Q}{2\pi}\frac{\vec{r}}{r^2}$ then $\nabla \bullet \vec{E} = \nabla \bullet \left[\frac{Q}{2\pi}\frac{\vec{r}}{r^2}\right] = \frac{Q}{2\pi}\nabla \bullet \frac{\vec{r}}{r^2} = Q\delta(\vec{r})$. Now once more apply Green's theorem to the Coulomb field. Use the divergence form of the theorem and this time appreciate that the divergence of $\vec{E}$ is not strictly zero, rather, the dirac-delta function captures the divergence: recall the RHS of this calculation followed from direct calculation of the flux of the Coloumb field through the circle ∂R,

$$\oint_{\partial R} \vec{E} \bullet \vec{n}\, ds = \iint_R \nabla \bullet \vec{E}\, dA \quad \Rightarrow \quad Q = \iint_R Q\delta(\vec{r})dA = Q.$$

All is well. This is the way to extend Green's theorem for Coulomb fields. You might wonder about other types of singularities. Are there similar techniques? Probably, but that is beyond these notes. I merely wish to sketch the way we think about these issues in electrostatics. In truth, this section is a bit of a novelty. What really matters is three-dimensional Coulomb fields whose magnitude depends on the squared-reciprocal of the distance from source charge to field point. Perhaps I will write an analogus section once we have developed the concepts of flux and the three-dimensional divergence theorem.

[16] I don't intend to explain where this 2π comes from, except to tell you that it must be there in order for the extension of Green's theorem to work out nicely.