LECTURE 35: SURFACE INTEGRALS (OF VECTOR FIELDS)

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7.6 surface integrals

We discuss how to integrate over surfaces in $\mathbb{R}^3$. This section is the natural extension of the work we have already accomplished with curves. We begin by describing the integral with respect to the surface area, naturally this integral calculates surface area, but more generally it allows us to continuously sum some density over a given surface. Next we discuss how to find the flux through a surface. The concept of flux requires we give the surface a direction. The flux of a vector field is the *number*[17] of field lines which cut through the surface. The parametric viewpoint is primary in this section, but we also make an effort to show how to calculate surface integrals from the graphical or level-surface viewpoint.

7.6.1 motivations for surface area integral and flux

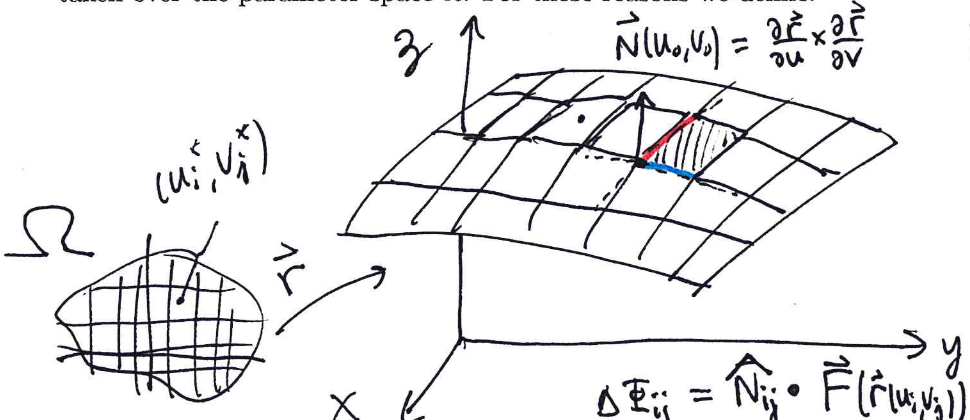
Let us consider a surface S parametrized by smooth $\vec{r}(u,v) = \langle x(u,x), y(u,v), z(u,v)\rangle$ for $(u,v) \in \Omega$. If we wish to approximate the surface area of S then we should partition the parameter space Ω into subregions Ω_{ij} such that $\bigcup_{i=1}^{m}\bigcup_{j=1}^{m}\Omega_{ij} = \Omega$ where we assume that Ω_{ij} are mostly disjoint, they might share an edge or point, but not an area. Naturally this partitions S into subsurfaces; $S = \bigcup_{i=1}^{m}\bigcup_{j=1}^{m} S_{ij}$ where $S_{ij} = \vec{r}(\Omega_{ij})$. Next, we replace each subsurface S_{ij} with its tangent plane based at some point[18] $\vec{r}(u_i^*, v_j^*) \in S_{ij}$. For convenience of this motivation we may assume that the partition is made so that Ω_{ij} is a little rectangle which is Δu by Δv. The length of the coordinate curves are well-approximated by $\frac{\partial \vec{r}}{\partial u}\Delta u$ and $\frac{\partial \vec{r}}{\partial v}\Delta v$ based at the point $\vec{r}(u_i^*, v_j^*)$. The coordinate lines on S are not necessarily perpendicular so the area is not simply the product of length times width, in fact, we have a little paralellogram to consider. It follows that the area A_{ij} of the i,j-tangent plane is given by

$$A_{ij} = \left|\left|\left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right)(u_i^*, v_j^*)\right|\right| \Delta u \Delta v.$$

A good approximation to surface area is given for $m, n >> 1$ by

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left|\left|\left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right)(u_i^*, v_j^*)\right|\right| \Delta u \Delta v$$

As we pass to the limit $m \to \infty$ and $n \to \infty$ the double finite sum becomes the double integral taken over the parameter space Ω. For these reasons we define:



$\vec{r}(u_0, v_0)$

$u \mapsto \vec{r}(u, v_0)$

$v \mapsto \vec{r}(u_0, v)$

$\frac{\partial \vec{r}}{\partial u}(u_0,v_0)\Delta u = l_u$

$\frac{\partial \vec{r}}{\partial v}(u_0,v_0)\Delta v = l_v$

[17] relative to some convention, lines drawn per unit of flux

[18] when I illustrate this idea I usually take these points to the lower left of the partition region, but in principle you could sample elsewhere.

Definition 7.6.1. *scalar surface integral.*

> Suppose S is a surface in $\mathbb{R}^3$ paramterized by $\vec{r}$ with domain Ω and with parameters u, v. Furthermore, suppose f is a continuous function on some open set containing S then we define the scalar surface integral of f over S by the following integral (when it exists)
>
> $$\iint_S f\,dS = \iint_\Omega f(\vec{r}(u,v)) \left|\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right|\right| du\,dv.$$

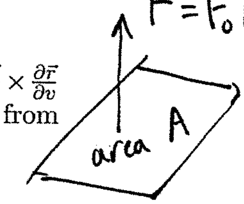
In the case we integrate $f = 1$ over S then we obtain the **surface area** of S. On the other hand, if f was the mass-density $f = \frac{dM}{dS}$ then $dM = f dS$ and the integral $\iint_S f\,dS$ calculates the total mass of S. Clearly it is convenient to think of dS as something on its own, however, it should be remembered that this is just notation to package the careful definition given above,

$\vec{A} = A\hat{n}$

$$\boxed{dS = \left|\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right|\right| du\,dv}$$ **infinitesimal scalar surface area element**

$= N\,du\,dv$ where $\vec{N} = \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = N\hat{N}$

$\vec{F} = F_0\hat{n}$



Moreover, we should recall that the normal vector field to S induced by $\vec{r}$ was given by $\vec{N} = \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$ hence we can write $dS = N du dv$. This is nice, but for most examples it does not save us from explicit calculation of the cross-product.

Next, consider a vector field $\vec{F}$ defined on some open set containing S. Suppose that S is a regular surface and as such has a well-defined normal vector field $\vec{N}$. If we define $\hat{n} = \frac{1}{N}\vec{N}$ then $\vec{F} \bullet \hat{n}$ gives the component of $\vec{F}$ which points in the normal direction of the surface. It is customary to draw field-lines to illustrate both the direction and magnitude of a vector field. The number of lines crossing a particular surface illustrates the magnitude of the vector field relative to the given area. For example, if we had an area A_1 which had 4 field lines of $\vec{F}$ and another area A_2 which had 8 field lines of $\vec{F}$ then the magnitude of $\vec{F}$ on these areas is proportional to $\frac{4}{A_1}$ and $\frac{8}{A_2}$ respectively. If the vector fields are constant, then the flux through A_1 is $F_1 A_1$ whereas the flux through A_2 is $F_2 A_2$. Generally, the flux of a vector field through a surface depends both on the size of the surface and the magnitude of the vector field in the normal direction of the surface. This is the natural generalization of the flux-integral we discussed previously for curves in the plane.

$\Phi = F_0 A$

$\Phi = \vec{F} \bullet \vec{A}$

Definition 7.6.2. *surface integral of vector field; the flux through a surface.*

> Suppose S is an oriented surface in $\mathbb{R}^3$ paramterized by $\vec{r}$ with domain Ω and with parameters u, v which induce unit-normal vector field $\hat{n}$. Furthermore, suppose $\vec{F}$ is a continuous vector field on some open set containing S then we define the **surface integral of** $\vec{F}$ over S by the following integral (when it exists)
>
> $$\iint_S \vec{F} \bullet d\vec{S} = \iint_S (\,\vec{F} \bullet \hat{n}\,) dS$$

In practice, the formula we utilize for direct computation is not the one given above. Let us calculate,

$$\iint_S (\,\vec{F} \bullet \hat{n}\,) dS = \iint_\Omega (\,\vec{F} \bullet \frac{1}{N}\vec{N}\,) N\,du\,dv = \boxed{\iint_\Omega (\,\vec{F} \bullet \vec{N}\,)\,du\,dv.}$$

Hence, recalling once more that $\vec{N}(u,v) = \partial_u \vec{r} \times \partial_v \vec{r}$ we find

$$\boxed{\iint_S \vec{F} \bullet d\vec{S} = \iint_\Omega \vec{F}(\vec{r}(u,v)) \bullet \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right) du\, dv.}$$

← alternate Def^{n} for flux of $\vec{F}$ through $\vec{S}$.

The equation boxed above is how we typically calculate flux of $\vec{F}$ through S. I should mention that is often convenient to calculate $d\vec{S}$ separately before computation of the integral, this quantity is called the **infinitesimal vector surface area element** and is defined by

$$\boxed{d\vec{S} = \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} du\, dv = \hat{n} dS} \qquad \textbf{vector area element } d\vec{S}$$

$d\vec{S} = \vec{N}\, du dv$
$= N \hat{n}\, du dv$
$= \hat{n}\, N\, du dv = \hat{n}\, dS.$

Once more, dS is the scalar area element. Both of these are only meaningful when viewed in connection with the parametric set-up described in this section.

The uninterested reader may skip to the examples, however, there is some unfinished business theoretically here. We should demonstrate that the definitions given in this section are independent of the parametrization. If this fails to be true then the concepts of surface area, total mass etc... and flux are in doubt. We must consider a **reparametrization** of S by $\vec{X} : D \to S$ where a, b are the typical parameters in D and the normal vector field induced by $\vec{X}$ is $\vec{N}_X(a,b) = \partial_a \vec{X} \times \partial_b \vec{X}$. Let us, in contrast denote $\vec{N}_r(u,v) = \partial_u \vec{r} \times \partial_v \vec{r}$. Since each point on S is covered smoothly by both $\vec{r} = \langle x, y, z \rangle$ and $\vec{X} = \langle X_1, X_2, X_3 \rangle$ there exist functions which transition between the two parametrizations. In particular, we can find $\vec{T} : \Omega \to D$ such that $\vec{T} = \langle h, g \rangle$ and

$$\vec{r}(u,v) = \vec{X}(h(u,v), g(u,v))$$

We need to sort through the partial derivatives so we can understand how the normal vector fields $\vec{N}_X$ and $\vec{N}_r$ are related, let $a = h(u)$ and $b = g(v)$ hence $\vec{r}(u,v) = \vec{X}(a,b)$. I'll expand $\vec{X}$ into its component function notation to make sure we understand what we're doing here:

$$\frac{\partial \vec{r}}{\partial u} = \frac{\partial}{\partial u} \Big\langle X_1(a,b), X_2(a,b), X_3(a,b) \Big\rangle = \Big\langle \frac{\partial}{\partial u}[X_1(a,b)], \frac{\partial}{\partial u}[X_2(a,b)], \frac{\partial}{\partial u}[X_3(a,b)] \Big\rangle$$

We calculate by the chain-rule, (omitting the (a,b) dependence on the lhs, technically we should write $\frac{\partial}{\partial u}[X_1(a,b)]$ etc...)

$$\frac{\partial X_1}{\partial u} = \frac{\partial X_1}{\partial a}\frac{\partial h}{\partial u} + \frac{\partial X_1}{\partial b}\frac{\partial g}{\partial u}, \quad \frac{\partial X_2}{\partial u} = \frac{\partial X_2}{\partial a}\frac{\partial h}{\partial u} + \frac{\partial X_2}{\partial b}\frac{\partial g}{\partial u}, \quad \frac{\partial X_3}{\partial u} = \frac{\partial X_3}{\partial a}\frac{\partial h}{\partial u} + \frac{\partial X_3}{\partial b}\frac{\partial g}{\partial u}$$

Likewise, for the derivative with respect to v we calculate,

$$\frac{\partial X_1}{\partial v} = \frac{\partial X_1}{\partial a}\frac{\partial h}{\partial v} + \frac{\partial X_1}{\partial b}\frac{\partial g}{\partial v}, \quad \frac{\partial X_2}{\partial v} = \frac{\partial X_2}{\partial a}\frac{\partial h}{\partial v} + \frac{\partial X_2}{\partial b}\frac{\partial g}{\partial v}, \quad \frac{\partial X_3}{\partial v} = \frac{\partial X_3}{\partial a}\frac{\partial h}{\partial v} + \frac{\partial X_3}{\partial b}\frac{\partial g}{\partial v}$$

We find,

$$\begin{aligned}
\frac{\partial \vec{r}}{\partial u} &= \left\langle \frac{\partial X_1}{\partial a}\frac{\partial h}{\partial u} + \frac{\partial X_1}{\partial b}\frac{\partial g}{\partial u}, \frac{\partial X_2}{\partial a}\frac{\partial h}{\partial u} + \frac{\partial X_2}{\partial b}\frac{\partial g}{\partial u}, \frac{\partial X_3}{\partial a}\frac{\partial h}{\partial u} + \frac{\partial X_3}{\partial b}\frac{\partial g}{\partial u} \right\rangle \\
&= \left\langle \frac{\partial X_1}{\partial a}, \frac{\partial X_2}{\partial a}, \frac{\partial X_3}{\partial a} \right\rangle \frac{\partial h}{\partial u} + \left\langle \frac{\partial X_1}{\partial b}, \frac{\partial X_2}{\partial b}, \frac{\partial X_3}{\partial b} \right\rangle \frac{\partial g}{\partial u} \\
&= \frac{\partial \vec{X}}{\partial a}\frac{\partial h}{\partial u} + \frac{\partial \vec{X}}{\partial b}\frac{\partial g}{\partial u}
\end{aligned}$$

Similarly,

$$\frac{\partial \vec{r}}{\partial v} = \frac{\partial \vec{X}}{\partial a}\frac{\partial h}{\partial v} + \frac{\partial \vec{X}}{\partial b}\frac{\partial g}{\partial v}$$

Calculate, by the antisymmetry of the cross-product,

$$\begin{aligned}
\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} &= \left(\frac{\partial \vec{X}}{\partial a}\frac{\partial h}{\partial u} + \frac{\partial \vec{X}}{\partial b}\frac{\partial g}{\partial u}\right) \times \left(\frac{\partial \vec{X}}{\partial a}\frac{\partial h}{\partial v} + \frac{\partial \vec{X}}{\partial b}\frac{\partial g}{\partial v}\right) \\
&= \left[\frac{\partial \vec{X}}{\partial a} \times \frac{\partial \vec{X}}{\partial b}\right]\frac{\partial h}{\partial u}\frac{\partial g}{\partial v} + \left[\frac{\partial \vec{X}}{\partial b} \times \frac{\partial \vec{X}}{\partial a}\right]\frac{\partial g}{\partial u}\frac{\partial h}{\partial v} \\
&= \left[\frac{\partial \vec{X}}{\partial a} \times \frac{\partial \vec{X}}{\partial b}\right]\left[\frac{\partial h}{\partial u}\frac{\partial g}{\partial v} - \frac{\partial g}{\partial u}\frac{\partial h}{\partial v}\right]
\end{aligned}$$

It follows that $\vec{N}_r(u,v) = \vec{N}_X(h(u,v), g(u,v))\frac{\partial(h,g)}{\partial(u,v)}$ hence $N_r(u,v) = N_X(h(u,v), g(u,v))\left|\frac{\partial(h,g)}{\partial(u,v)}\right|$. The vertical bars denote absolute value; when we pull a scalar out of a magnitude of a vector it gets absolute value bars; $||c\vec{v}|| = |c|\,||\vec{v}||$. Consider the surface area integral of f over S which is parametrized by both $\vec{r}(u,v)$ and $\vec{X}(a,b)$ as discussed above. Observe,

$$\begin{aligned}
\iint_S f\, dS &= \iint_\Omega f(\vec{r}(u,v))N_r(u,v)du\, dv \\
&= \iint_\Omega f(\vec{X}(h(u,v), g(u,v)))N_X(h(u,v), g(u,v))\left|\frac{\partial(h,g)}{\partial(u,v)}\right|du\, dv \\
&= \iint_D f(\vec{X}(a,b))N_X(a,b)da\, db.
\end{aligned}$$

In the last line I applied the multivariate change of variables theorem. Notice that the absolute value bars are important to the calculation. We will see in the corresponding calculation for flux the absolute value bars are absent, but this is tied to the orientation-dependence of the flux integral. In the scalar surface integral the direction (outward or inward) of the normal vector field does not figure into the calculation. We have shown

$$\boxed{\iint_S f\, dS = \iint_{-S} f\, dS}.$$

Next, turn to the reparametrization invariance of the flux integral. Suppose once more $\vec{r}$ and $\vec{X}$ both parametrize S. Calculate the flux of a continuous vector field $\vec{F}$ defined on some open set containing S (via the boxed equation following the definition)

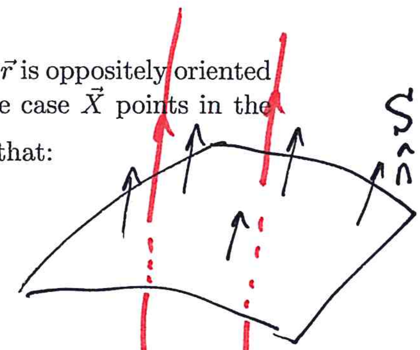
$$\begin{aligned}
\iint_S \vec{F} \bullet d\vec{S} &= \iint_\Omega \vec{F}(\vec{r}(u,v)) \bullet \vec{N}_r(u,v)\, du\, dv \\
&= \iint_\Omega \vec{F}(\vec{X}(h(u,v), g(u,v))) \bullet \vec{N}_X(h(u,v), g(u,v))\frac{\partial(h,g)}{\partial(u,v)}\, du\, dv
\end{aligned}$$

If $\frac{\partial(h,g)}{\partial(u,v)} > 0$ for all $(u,v) \in \Omega$ then it follows that $\vec{N}_r$ and $\vec{N}_X$ point on the same side of S and we say they are **consistently oriented** parametrizations of S. Clearly if we wish for the flux to be meaningful we must choose a side for S and insist that we use a consistently oriented parametrization to calculate the flux. If $\vec{r}$ and $\vec{X}$ are consistently oriented then $\frac{\partial(h,g)}{\partial(u,v)} > 0$ for all

$(u, v) \in \Omega$ and hence $\left|\frac{\partial(h,g)}{\partial(u,v)}\right| = \frac{\partial(h,g)}{\partial(u,v)}$ for all $(u, v) \in \Omega$ and we find

$$\begin{aligned}\iint_S \vec{F} \bullet d\vec{S} &= \iint_\Omega \vec{F}(\vec{X}(h(u,v), g(u,v))) \bullet \vec{N}_X(h(u,v), g(u,v)) \left|\frac{\partial(h,g)}{\partial(u,v)}\right| du\,dv \\ &= \iint_D \vec{F}(\vec{X}(a,b) \bullet \vec{N}_X(a,b)\,da\,db\end{aligned}$$

by the change of variable theorem for double integrals. On the other hand, if $\vec{r}$ is oppositely oriented from $\vec{X}$ then we say $\vec{r}$ parametrizes S whereas $\vec{X}$ parametrizes $-S$. In the case $\vec{X}$ points in the direction opposite $\vec{r}$ we find the coefficient $\left|\frac{\partial(h,g)}{\partial(u,v)}\right| = -\frac{\partial(h,g)}{\partial(u,v)}$ and it follows that:



$$\boxed{\iint_{-S} \vec{F} \bullet d\vec{S} = -\iint_S \vec{F} \bullet d\vec{S}}$$

7.6.2 standard surface examples

In this section we derive $d\vec{S}$ and dS for the sphere, cylinder, cone, plane and arbitrary graph. You can add examples past these, but these are essential. I also derive the surface area where appropriate.

1. S_R the **sphere of radius** R centered at the origin. We have spherical equation $\rho = R$ which suggests the natural parametric formulas: for $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$

$$x = R\cos(\theta)\sin(\phi), \qquad y = R\sin(\theta)\sin(\phi), \qquad z = R\cos(\phi)$$

$$\vec{r}(\phi, \theta) = \langle R\cos(\theta)\sin(\phi),\ R\sin(\theta)\sin(\phi),\ R\cos(\phi)\rangle$$

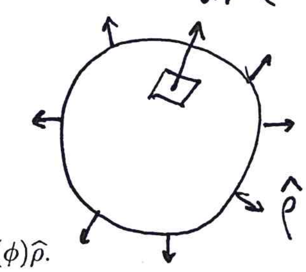
$\hat{\rho}$

We can limit the parameter space $[0, \pi] \times [0, 2\pi]$ to select subsets of the sphere if need arises. The normal vector field is calculated from partial derivatives of $\vec{r}(\phi, \theta)$;

$d\vec{S} = (R^2 \sin\phi d\phi d$

$$\frac{\partial \vec{r}}{\partial \phi} = \Big\langle R\cos(\theta)\cos(\phi),\ R\sin(\theta)\cos(\phi),\ -R\sin(\phi)\Big\rangle$$

$$\frac{\partial \vec{r}}{\partial \theta} = \Big\langle -R\sin(\theta)\sin(\phi),\ R\cos(\theta)\sin(\phi),\ 0\Big\rangle$$



In invite the reader to calculate the cross-product above and derive that

$$\frac{\partial \vec{r}}{\partial \phi} \times \frac{\partial \vec{r}}{\partial \theta} = R^2\sin(\phi)\Big\langle \cos(\theta)\sin(\phi),\ \sin(\theta)\sin(\phi),\ \cos(\phi)\Big\rangle = R^2\sin(\phi)\hat{\rho}.$$

We will find is useful to note that $\vec{N}(\phi, \theta) = R^2\sin(\phi)\hat{\rho}$ for S_R. This is an outward pointing normal vector field. To summarize, for the sphere S_R with outward orientation we find

$$\boxed{d\vec{S} = R^2\sin(\phi)d\phi d\theta\hat{\rho} \qquad \text{and} \qquad dS = R^2\sin(\phi)d\phi d\theta}$$

The surface area of the sphere S_R is given by

$$Area(S_R) = \int_0^{2\pi}\int_0^{\pi} R^2\sin(\phi)d\phi d\theta = 4\pi R^2. \quad = \frac{d}{dR}\left(\frac{4}{3}\pi R^3\right)$$

It is interesting to note that $d/dR(\frac{4}{3}\pi R^3) = 4\pi R^2$ just like $d/dR(\pi R^2) = 2\pi R$.

2. **Right circular cylinder** of radius R with axis along z. In cylindrical coordinates we have the simple formulation $r = R$ which gives the natural parametrization:

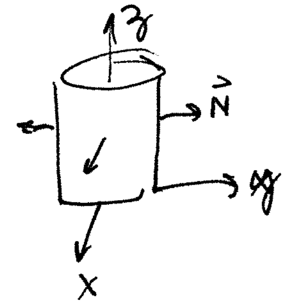
$$x = R\cos(\theta), \qquad y = R\sin(\theta), \qquad z = z$$

$$\vec{r}(\theta, z) = \langle R\cos(\theta),\ R\sin(\theta),\ z\rangle$$

for $0 \leq \theta \leq 2\pi$ and $z \in \mathbb{R}$. Calculate $\frac{\partial \vec{r}}{\partial \theta} = \langle -R\sin(\theta),\ R\cos(\theta), 0\rangle = R\widehat{\theta}$ and $\frac{\partial \vec{r}}{\partial z} = \widehat{z}$ thus

$$\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial z} = R\widehat{\theta} \times \widehat{z} = R\widehat{r} \qquad \vec{N}(\theta,z) = \langle R\cos\theta, R\sin\theta, 0\rangle$$

Consequently, $\vec{N}(\theta, z) = R\widehat{r} = R\langle\cos(\theta), \sin(\theta), 0\rangle$ and we find



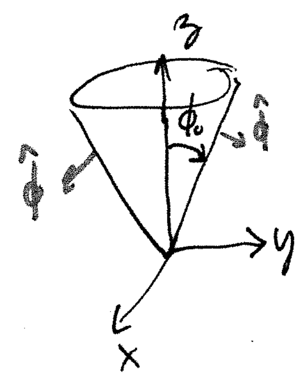
$$\boxed{d\vec{S} = R d\theta\, dz\, \widehat{r} \qquad \text{and} \qquad dS = R d\theta dz}$$

The surface area of the cylinder for $0 \leq z \leq L$ is given by

$$Area = \int_0^{2\pi}\int_0^L R dz\, d\theta = 2\pi RL.$$

Of course, the whole cylinder with unbounded z has infinite surface area.

3. **Cone** at angle ϕ_o. In cylindrical coordinates $r = \rho\sin(\phi_o)$ thus the cartesian equation of this cone is easily derived from $r^2 = \rho^2\sin^2(\phi_o)$ gives $x^2 + y^2 = \sin^2(\phi_o)(x^2 + y^2 + z^2)$ hence, for $\phi_o \neq \pi/2$, we find $x^2 + y^2 = \tan^2(\phi_o)z^2$. In cylindrical coordinates this cone has equation $r = \tan(\phi_o)z$. From spherical coordinates we find a natural parametrization,



$$x = \rho\cos(\theta)\sin(\phi_o), \qquad y = \rho\sin(\theta)\sin(\phi_o), \qquad z = \rho\cos(\phi_o)$$

For convenience denote $a = \sin(\phi_o)$ and $b = \cos(\phi_o)$ thus

$$\vec{r}(\theta, \rho) = \langle a\rho\cos(\theta),\ a\rho\sin(\theta),\ b\rho\rangle$$

for $0 \leq \theta \leq 2\pi$ and $\rho \in [0, \infty)$. Differentiate to see that

$$\frac{\partial \vec{r}}{\partial \theta} = \langle -a\rho\sin(\theta),\ a\rho\cos(\theta),\ 0\rangle \qquad \& \qquad \frac{\partial \vec{r}}{\partial \rho} = \langle a\cos(\theta),\ a\sin(\theta),\ b\rangle.$$

Calculate,

$$\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \rho} = \langle ab\rho\cos(\theta), ab\rho\sin(\theta), -a^2\rho\rangle = a\rho\langle b\cos(\theta), b\sin(\theta), -a\rangle$$

Note that $\widehat{\phi} = \langle\cos(\theta)\cos(\phi), \cos(\theta)\cos(\phi), -\sin(\phi)\rangle$ and $a = \sin(\phi_o)$ and $b = \cos(\phi_o)$ hence

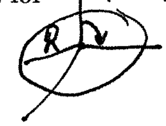
$$\vec{N}(\theta,\rho) = \frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \rho} = \rho\sin(\phi_o)\widehat{\phi}$$

Consequently, $\vec{N}(\theta, \rho) = \rho\sin(\phi_o)\widehat{\phi}$. We find for the cone $\phi = \phi_o$,

$$\boxed{d\vec{S} = \rho\sin(\phi_o)d\theta\, d\rho\widehat{\phi} \qquad \text{and} \qquad dS = \rho\sin(\phi_o)d\theta\, d\rho.}$$

where $\widehat{\phi} = \langle \cos(\theta)\cos(\phi_o), \cos(\theta)\cos(\phi_o), -\sin(\phi_o)\rangle$. The surface area of the cone $\phi = \phi_o$ for $0 \leq \rho \leq R$ is given by

$\phi_o = \pi/2$



$$Area = \int_0^R \int_0^{2\pi} \rho \sin(\phi_o) d\theta\, d\rho = \sin(\phi_o)\pi R^2.$$

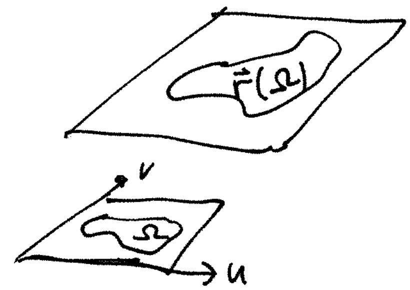
Of course, the whole cone with unbounded ρ has infinite surface area. On the other hand, the result above is quite reasonable in the case $\phi_o = \pi/2$.

4. **Plane containing vectors $\vec{A}$ and $\vec{B}$ and base-point $\vec{r}_o$.** We parametrize by

$$\vec{r}(u,v) = \vec{r}_o + u\vec{A} + v\vec{B}$$

for $(u,v) \in \mathbb{R}^{\,n\times n}$. Clearly $\partial_u \vec{r} = \vec{A}$ and $\partial_v \vec{r} = \vec{B}$ hence $\vec{N}(u,v) = \vec{A} \times \vec{B}$. The plane has a constant normal vector field. We find:

$$\boxed{d\vec{S} = \vec{A} \times \vec{B} du\, dv \qquad \text{and} \qquad dS = ||\vec{A} \times \vec{B}|| du\, dv.}$$



If we select a compact subset Ω of the plane then $\vec{r}(\Omega)$ has surface area

$$Area = \iint_\Omega ||\vec{A} \times \vec{B}|| du\, dv = \iint_\Omega ||\vec{A} \times \vec{B}||\, dA.$$

$$= ||\vec{A} \times \vec{B}|| \iint dA$$

In the last equation I mean to emphasize that the problem reduces to an ordinary double integral of a constant over the parameter space Ω. Usually there is some parameter dependence in dS, but the plane is a very special case.

$Area = ||\vec{A} \times \vec{B}||\, area(\Omega)$

5. **Graph** $z = f(x,y)$. Naturally we take parameters x, y and form

$$\vec{r}(x,y) = \langle x, y, f(x,y)\rangle$$

this is the *Monge patch* on the surface formed by the graph. Differentiate,

$$\partial_x \vec{r} = \langle 1, 0, \partial_x f\rangle \qquad \& \qquad \partial_y \vec{r} = \langle 0, 1, \partial_y f\rangle$$

Calculate the normal vector field,

$$\vec{N}(x,y) = \langle 1, 0, \partial_x f\rangle \times \langle 0, 1, \partial_y f\rangle = \langle -\partial_x f, -\partial_y f, 1\rangle$$

We find:

$$\boxed{d\vec{S} = \langle -\partial_x f, -\partial_y f, 1\rangle\, dx\, dy \qquad \text{and} \qquad dS = \sqrt{1 + (\partial_x f)^2 + (\partial_y f)^2}\, dx\, dy.}$$

If Ω is a compact subset of $dom(f)$ then we can calculate the surface area by

$$Area = \iint_\Omega \sqrt{1 + (\partial_x f)^2 + (\partial_y f)^2}\, dx\, dy.$$

7.6.3 scalar surface integrals

I examine several examples to further illustrate the construction of the scalar surface integral. We use the surface area elements developed in the previous section.

half x>0

Example 7.6.3. *Calculate the total mass of a sphere of radius R which has mass-density $\sigma(x,y,z) = \tan^{-1}(y/x)$. Identify that $\sigma = dM/dS$ hence $dM = \sigma dS$. Moreover, in spherical coordinates $\sigma = \theta$ hence thus the integral below gives the total mass:*

omit

$$M = \iint_{S_R} \sigma\, dS = \int_{\cancel{0}\,-\pi/2}^{\cancel{2\pi}\,\pi/2} \int_0^{\pi} R^2\theta \sin(\phi)d\phi d\theta = R^2\Big(-\cos(\phi)\Big|_0^{\pi}\Big)\Big(\frac{\theta^2}{2}\Big|_{\cancel{0}\,\pi/2}^{\cancel{2\pi}\,-\pi/2}\Big) = \boxed{4\pi^2R^2.}$$

Example 7.6.4. *Calculate the average of $f(x,y,z) = z^2$ over the circular cylinder S: $x^2+y^2 = R^2$ for $0 \leq z \leq L$ (assume the caps are open, just find the average over the curved side). By logic,*

$$f_{avg} = \frac{1}{2\pi RL}\iint_S z^2 dS = \frac{1}{2\pi RL}\int_0^L \int_0^{2\pi} Rz^2 d\theta\, dz = \frac{1}{2\pi RL}(2\pi R)(L^3/3) = \boxed{\frac{L^2}{3}.}$$

Generally if we wish to calculate the average of a function over a surface of finite total surface area we define f_{avg} to be the value such that $\iint_S f\, dS = f_{avg}Area(s)$.

Example 7.6.5. *Find the centroid of the cone $\phi = \pi/4$ for $0 \leq \rho \leq R$. The centroid is the geometric center of the object with regard to the density. In other words, calculate the center of mass under the assumption $dM/dS = 1$. However you like to think of it, the centroid $(\bar{x}, \bar{y}, \bar{z})$ is given by*

$$\bar{x} = \frac{1}{Area(S)}\iint_S x dS, \qquad \bar{y} = \frac{1}{Area(S)}\iint_S y\, dS, \qquad \bar{z} = \frac{1}{Area(S)}\iint_S z\, dS$$

For the cone S it is clear by symmetry that $\bar{x} = \bar{y} = 0$. Once more building off (3.) of the previous section we calculate: $Area(S) = \sin(\phi_o)\pi R^2 = \pi R^2/\sqrt{2}$ hence as $z = \rho\cos(\phi_o) = \rho/\sqrt{2}$ and $dS = \frac{\rho}{\sqrt{2}}d\theta\, d\rho$

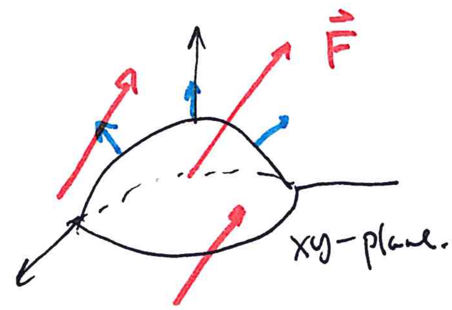
$$\bar{z} = \frac{\sqrt{2}}{\pi R^2}\int_0^R \int_0^{2\pi} \frac{\rho}{\sqrt{2}}\frac{\rho}{\sqrt{2}}d\theta\, d\rho = \frac{\sqrt{2}}{\pi R^2}\frac{R^3}{3}\frac{2\pi}{2} = \boxed{\frac{R\sqrt{2}}{3}.}$$

We can also calculate the moment of intertia about the z-axis for the cone S (assume constant mass density 1 for this example) . The moment of intertia is defined by $I_z = \iint_S r^2 dM$ and as the equation of this cone is simply $r = z$ we find $r^2 = \rho^2\cos^2(\pi/4) = \rho^2/2$ thus

$$I_z = \iint_S r^2\delta\, dS = \int_0^R \int_0^{2\pi} \frac{\rho^2}{2}\frac{\rho}{\sqrt{2}}d\theta\, d\rho = \frac{R^4(2\pi)}{8\sqrt{2}} = \boxed{\frac{\pi R^4}{4\sqrt{2}}.}$$

Example 7.6.6. *Find the scalar surface integral of $f(x,y,z) = xyz$ on the graph S: $z = 6+x+y$ for $0 \leq y \leq x^2$ and $0 \leq x \leq 1$ (this is just a portion of the total graph $z = 6+x+y$ which is an* *unbounded plane). Observe that $f_x = 1$ and $f_y = 1$ thus $dS = \sqrt{1+f_x^2+f_y^2}\,dx\,dy = \sqrt{3}\,dx\,dy$*

$$\begin{aligned}
\iint_S xyz\,dS &= \int_0^1\int_0^{x^2} xy(6+x+y)\sqrt{3}\,dy\,dx \\
&= \sqrt{3}\int_0^1\int_0^{x^2}\left(\,6xy+6x^2y+6xy^2\,\right)dy\,dx \\
&= \sqrt{3}\int_0^1\left(\,3xy^2+3x^2y^2+2xy^3\,\right)\Big|_0^{x^2}\,dx \\
&= \sqrt{3}\int_0^1\left(\,3x^5+3x^6+2x^7\,\right)\Big|_0^{x^2}\,dx \\
&= \sqrt{3}\left(\frac{3}{6}+\frac{3}{7}+\frac{2}{8}\right) \\
&= \boxed{\frac{33\sqrt{3}}{28}.}
\end{aligned}$$



7.6.4 flux integrals

Once more I build off the examples from Section 7.6.2.

Example 7.6.7. *Calculate the flux of $\vec{F} = \langle 1,2,3\rangle$ through the part of the sphere $x^2+y^2+z^2=4$ which is above the xy-plane. Recall $d\vec{S} = R^2\sin(\phi)d\theta\,d\phi\widehat{\rho}$ and note for $z \geq 0$ we need no restriction of the polar angle θ ($0 \leq \theta \leq 2\pi$) however the azmuthial angle ϕ falls into the interval $0 \leq \phi \leq \pi/2$. Thus, as $R = 2$ for this example,*

$$\begin{aligned}
\Phi = \iint_S \vec{F}\bullet d\vec{S} &= \int_0^{\pi/2}\int_0^{2\pi}\langle 1,2,3\rangle\bullet\big\langle\cos(\theta)\sin(\phi),\sin(\theta)\sin(\phi),\cos(\phi)\big\rangle 4\sin(\phi)d\theta\,d\phi \\
&= 4\int_0^{\pi/2}\int_0^{2\pi}\Big(\cancelto{0}{\cos(\theta)}\sin^2(\phi)+2\cancelto{0}{\sin(\theta)}\sin^2(\phi)+\underbrace{3\cos(\phi)\sin(\phi)}\Big)d\theta\,d\phi \\
&= 6\int_0^{\pi/2}\int_0^{2\pi}\Big(\sin(2\phi)\Big)d\theta\,d\phi \qquad 2\cos\phi\sin\phi = \sin(2\phi) \\
&= 12\pi\left(\frac{-1}{2}\cos(2\phi)\right)\Big|_0^{\pi/2} = 12\pi\Big(-\tfrac{1}{2}\underbrace{\cos(\pi)}_{-1}+\tfrac{1}{2}\cos(0)\Big) \\
&= \boxed{12\pi.}
\end{aligned}$$

$\frac{n-1}{2}$

Example 7.6.8. *Let $n \in \mathbb{Z}$ and calculate the flux of $\vec{F}(x,y,z) = (x^2+y^2+z^2)^{n/2-1}\langle x,y,z\rangle$ through the sphere S_R. Observe that $\vec{F} = \rho^n\widehat{\rho}$ and recall*

$\hat{\rho} = \dfrac{\langle x,y,z\rangle}{\sqrt{x^2+y^2+z^2}}$

$$\vec{r}(\phi,\theta) = \big\langle R\cos(\theta)\sin(\phi), R\sin(\theta)\sin(\phi), R\cos(\phi)\big\rangle = R\widehat{\rho}$$

Thus calculate, $\vec{F}(\vec{r}(\phi,\theta)) = R^n\widehat{\rho}$

$$\begin{aligned}
\Phi = \iint_S \vec{F}\bullet d\vec{S} &= \int_0^{\pi}\int_0^{2\pi}(\,R^n\,\widehat{\rho}\,)\bullet(\,\widehat{\rho}R^2\sin(\phi)d\theta\,d\phi) \\
&= R^{n+2}\int_0^{\pi}\int_0^{2\pi}\sin(\phi)d\theta\,d\phi \\
&= \boxed{4\pi R^{n+2}.}
\end{aligned}$$

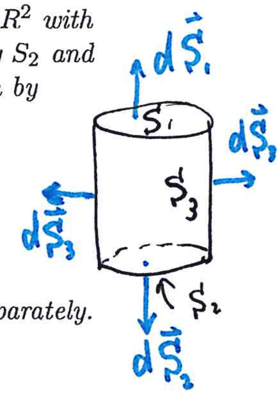
Coulomb Field

$n=-2$ is neat

$$\vec{F} = \frac{\langle x,y,z\rangle}{\left(\sqrt{x^2+y^2+z^2}\right)^3} = \frac{1}{\rho^2}\hat{\rho}$$

Believe it!. Notice that the case $n = -2$ is very special. In that case the flux is independent of the radius of the sphere. The flux spreads out evenly and is neither created nor destroyed for $n = -2$.

Example 7.6.9. *Calculate the flux of $\vec{F} = \langle x^2, z, y\rangle$ through the closed cylinder $x^2 + y^2 = R^2$ with $0 \leq z \leq L$. Notice that $S = S_1 \cup S_2 \cup S_3$ where I mean to denote the top by S_1, the base by S_2 and the side by S_3. The parametrizations and normal vectors to these faces are naturally given by*



$$\vec{X}_1(r,\theta) = \langle r\cos(\theta), r\sin(\theta), L\rangle \qquad \vec{N}_1(r,\theta) = \partial_r\vec{X}_1 \times \partial_\theta\vec{X}_1 = \widehat{r} \times r\widehat{\theta} = r\,\widehat{z}$$

$$\vec{X}_2(\theta,r) = \langle r\cos(\theta), r\sin(\theta), L\rangle \qquad \vec{N}_2(\theta,r) = \partial_\theta\vec{X}_2 \times \partial_r\vec{X}_2 = r\widehat{\theta} \times \widehat{r} = -r\,\widehat{z}$$

$$\vec{X}_3(\theta,z) = \langle R\cos(\theta), R\sin(\theta), z\rangle \qquad \vec{N}_3(\theta,z) = \partial_\theta\vec{X}_3 \times \partial_z\vec{X}_3 = R\widehat{\theta} \times \widehat{z} = R\widehat{r}$$

where $0 \leq r \leq R$, $0 \leq \theta \leq 2\pi$ and $0 \leq z \leq L$. I'll calculate the flux through each face separately. Begin with S_1:

$$\vec{F}(\vec{X}_1(r,\theta)) = \vec{F}(r\cos(\theta), r\sin(\theta), L) = \big\langle r^2\cos^2(\theta), L, r\sin(\theta)\big\rangle$$

note that $d\vec{S}_1 = r dr\, d\theta\, \widehat{z}$ and we find

$$\Phi_{S_1} = \iint_{S_1} \vec{F} \bullet d\vec{S} = \int_0^R \int_0^{2\pi} \langle r^2\cos^2(\theta), L, r\sin(\theta)\rangle \bullet \widehat{z}\big(r d\theta\, dr\big) = \int_0^R \int_0^{2\pi} r^2\sin(\theta) d\theta\, dr = 0.$$

Through a similar calculation we find $\Phi_{S_2} = 0$. To calculate the flux through S_3 we should evaluate the vector field on the parametrization,

$$\vec{F}(\vec{X}_3(r,\theta)) = \vec{F}(\overbrace{R\cos(\theta)}^{x}, \overbrace{R\sin(\theta)}^{y}, \overline{z}^{\,z}) = \big\langle R^2\cos^2(\theta), z, R\sin(\theta)\big\rangle \qquad \vec{F}(x,y,z) = \langle x^2, z, y\rangle$$

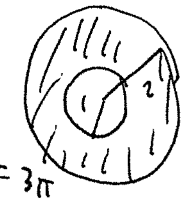
also recall that $\widehat{r} = \langle\cos(\theta), \sin(\theta), 0\rangle$ thus $d\vec{S} = R\langle\cos(\theta), \sin(\theta), 0\rangle d\theta\, dz$. Thus,

$$\begin{aligned}
\Phi_{S_3} &= \int_0^{2\pi}\int_0^L \big\langle R^2\cos^2(\theta), z, R\sin(\theta)\big\rangle \bullet R\big\langle\cos(\theta), \sin(\theta), 0\big\rangle d\theta\, dz \\
&= \int_0^{2\pi}\int_0^L \Big(R^3\cos^3(\theta) + zR\sin(\theta)\Big)\, d\theta\, dz \\
&= LR^3 \int_0^{2\pi} \cos^3(\theta)\, d\theta \\
&= LR^3 \int_0^{2\pi} [1 - \sin^2(\theta)]\cos(\theta)\, d\theta \\
&= LR^3 \int_{\sin(0)}^{\sin(2\pi)} \big[1 - u^2\big] du \\
&= \boxed{0.}
\end{aligned}$$

Example 7.6.10. *Find the flux of $\vec{F} = \vec{C}$ on the subset of the plane $\vec{r}(u,v) = \vec{r}_o + u\vec{A} + v\vec{B}$ defined by $1 \leq u^2 + v^2 \leq 4$. Denote $\Omega = dom(\vec{r})$. You can calculate $d\vec{S} = \vec{A} \times \vec{B}\, du\, dv$ hence*

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_\Omega \vec{C} \cdot (\vec{A} \times \vec{B})\, du\, dv = \vec{C} \cdot (\vec{A} \times \vec{B}) \iint_\Omega du\, dv = \boxed{15\pi \vec{C} \cdot (\vec{A} \times \vec{B}).}$$

3π



Since the area of Ω is clearly $\pi(4)^2 - \pi(1)^2 = 15\pi$. 3π

2^2

$\pi(2^2 - 1^1) = 3\pi$

Finally, we conclude by developing a standard formula which is the focus of flux calculations in texts such as Stewart's.

Example 7.6.11. *Find the flux of $\vec{F} = \langle P, Q, R \rangle$ through the upwards oriented graph S: $z = f(x,y)$ with domain Ω. We derived that $d\vec{S} = \langle -\partial_x f, -\partial_y f, 1 \rangle dx\, dy$ relative to the Monge parametrization $\vec{r}(x,y) = \langle x, y, f(x,y) \rangle$ for $(x,y) \in \Omega$. We calculate, from the definition,*

$$\begin{aligned} \iint_S \langle P, Q, R \rangle \cdot d\vec{S} &= \iint_\Omega \langle P, Q, R \rangle \cdot \langle -\partial_x f, -\partial_y f, 1 \rangle dx\, dy \\ &= \iint_\Omega \left(-P\frac{\partial f}{\partial x} - Q\frac{\partial f}{\partial y} + R \right) dx\, dy \end{aligned}$$

which is technically incorrect, we really mean the following:

$$\iint_S \langle P, Q, R \rangle \cdot d\vec{S} = \iint_\Omega \left(-P(x,y,f(x,y))\frac{\partial f}{\partial x} - Q(x,y,f(x,y))\frac{\partial f}{\partial y} + R(x,y,f(x,y)) \right) dA$$

For example, to calculate the flux of $\vec{F}(x,y,z) = \langle -x, -y, e^{x^2+y^2} \rangle$ on $z = x^2 + y^2$ for $0 \leq x^2 + y^2 \leq 1$ we calculate $\partial_x f = 2x$ and $\partial_y f = 2y$

$$\iint_S \langle -x, -y, e^{x^2+y^2} \rangle \cdot d\vec{S} = \iint_\Omega \left(-2x^2 - 2y^2 + e^{x^2+y^2} \right) dA$$

In polar coordinates Ω is described by $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$ so we calculate

$$\iint_S \langle -x, -y, e^{x^2+y^2} \rangle \cdot d\vec{S} = \int_0^{2\pi} \int_0^1 \left(-2r^2 + e^{r^2} \right) r dr\, d\theta = 2\pi \left[\frac{-r^4}{2} + \frac{1}{2}e^{r^2} \right] \Bigg|_0^1 = \boxed{\pi(e-2).}$$