LECTURE 36: STOKES' THEOREM

 • 365–374 of 2020 notes

7.7 stokes' theorem

We have already encountered a simple version of Stokes' Theorem in the two-dimensional context. Recall that

$$\oint_{\partial R} P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

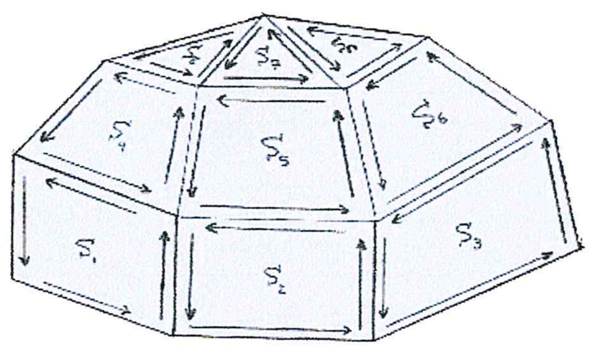
could be stated in terms of the z-component of the curl for $\vec{F} = \langle P, Q, 0 \rangle$:

$$\oint_{\partial R} \vec{F} \bullet d\vec{r} = \iint_R (\nabla \times \vec{F}) \bullet \widehat{z}\, dx\, dy$$

However, notice that the double integral above is actually the surface integral of $\nabla \times \vec{F}$ over the planar surface R where $d\vec{S} = \widehat{z} dx\, dy$. Let's generalize this idea a little. Suppose S is some simply connected planar region with unit-normal $\widehat{n}$ which is consistently oriented[19] with ∂S then we can derive Green's Theorem for S and by the arguments of the earlier section we have that

$$\oint_{\partial S} \vec{F} \bullet d\vec{r} = \iint_S (\nabla \times \vec{F}) \bullet \widehat{n}\, du\, dv = \iint_S (\nabla \times \vec{F}) \bullet d\vec{S}. \qquad \star$$

Suppose that $S = S_1 \cup S_2 \cup \cdots \cup S_n$ is a simply connected surface where each S_j is a planar region with unit normal $\widehat{n}$ and consistent boundary ∂S_j. The planar regions S_j are called **faces** and we call such a surface a **polyhedra**.



Theorem 7.7.1. *baby Stokes' for piece-wise flat surfaces.*

> Suppose S is a polyhedra S with consistently oriented boundary ∂S and suppose $\vec{F}$ is differentiable on some open set containing S then
>
> $$\oint_{\partial S} \vec{F} \bullet d\vec{r} = \iint_S (\nabla \times \vec{F}) \bullet d\vec{S}.$$

Proof: The proof is mostly by the picture before the theorem. The key is that because S is composed of flat faces we can apply $\star$ to each face and obtain for $j = 1, 2, \ldots, n$:

$$\oint_{\partial S_j} \vec{F} \bullet d\vec{r} = \iint_{S_j} (\nabla \times \vec{F}) \bullet d\vec{S}.$$

[19]this means as we travel around the boundary the surface is on the left

Add these equations together and identify the surface integral,

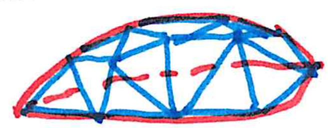
$$\sum_{j=1}^{n} \oint_{\partial S_j} \vec{F} \cdot d\vec{r} = \sum_{j=1}^{n} \iint_{S_j} (\nabla \times \vec{F}) \cdot d\vec{S} = \iint_{S_1 \cup S_2 \cup \cdots \cup S_n} (\nabla \times \vec{F}) \cdot d\vec{S}.$$

To simplify the sum of the circulations we need to realize that all the edges of faces which are interior cancel against oppositely oriented adjacent face edges. The only edge which leads to an uncancelled flow are those outer edges which are not common to two faces. This is best seen in the picture. It follows that

$$\sum_{j=1}^{n} \oint_{\partial S_j} \vec{F} \cdot d\vec{r} = \oint_{\partial S} \vec{F} \cdot d\vec{r}$$

which completes the proof of the theorem. □

The theorem above naturally extends to a theorem for mostly regular surfaces. I say *mostly* regular since we do allow for surfaces which have edges and corners. The normal vector field may vanish at such edges, however, it is assumed to be nonvanishing elsewhere. There are surfaces where the normal vector field vanishes at points other than the edge or corner. For example, the mobius band. Such a surface is **non-orientable**. Generally, we only wish to consider oriented surfaces. I implicitly assume S is oriented by stating it has consistently oriented boundary ∂S.

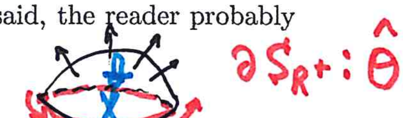


Theorem 7.7.2. *Stokes' Theorem for simply connected surface.*

> Suppose S is a simply connected surface S with consistently oriented boundary ∂S and suppose $\vec{F}$ is differentiable on some open set containing S then
>
> $$\oint_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}.$$

Proof: If S is a mostly regular, simply connected, surface then it can be approximately modeled by a simply connected polyhedra with n-faces. As we take $n \to \infty$ this approximation becomes exact and we obtain Stokes' Theorem. □.

The reader should find the limit above geometrically obvious, but analytically daunting. We will not pursue the full analysis of the limit imiplicit within the proof above. However, we will offer another proof of Stokes' Theorem for a curved surface of a simple type at the end of this section. This should help convince the reader of the generality of the theorem. That said, the reader probably just wants to see some examples at this point:



Example 7.7.3. *Let $\vec{F} = \langle -y, x, z \rangle$. Find the flux of $\nabla \times \vec{F}$ over the half of the outward-oriented sphere $\rho = R$ with $z \geq 0$. Denote the hemisphere S_{R+}. The hemisphere is simply connected and the boundary of the outward-oriented hemisphere is given by $x = R\cos(\theta)$, $y = R\sin(\theta)$ and $z = 0$.*

$\vec{F} = \langle -y, x, z \rangle$

$$\left.\begin{array}{l} x = R\cos\theta \\ y = R\sin\theta \\ z = 0 \end{array}\right\} 0 \leq \theta \leq 2\pi$$

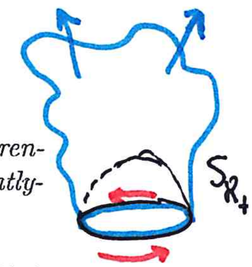
Apply Stokes' Theorem:

$$\iint_{S_{R+}} (\nabla \times \vec{F}) \bullet d\vec{S} = \oint_{\partial S_{R+}} \vec{F} \bullet d\vec{r}$$
$$= \int_0^{2\pi} \langle -R\sin(\theta), R\cos(\theta), 0 \rangle \bullet \langle -R\sin(\theta), R\cos(\theta), 0 \rangle d\theta$$
$$= \int_0^{2\pi} R^2 d\theta$$
$$= \boxed{2\pi R^2.}$$

$\partial S_{R-} = -\partial S_{R+}$

Example 7.7.4. *Let $\vec{F} = \langle -y, x, z \rangle$. Find the flux of $\nabla \times \vec{F}$ over the half of the outward-oriented sphere $\rho = R$ with $z < 0$. Denote the lower hemisphere by S_{R-}. To solve this we can use the result of the previous problem. Notice that S_{R+} and S_{R-} share the same set of points as a boundary, however, $\partial S_{R+} = -\partial S_{R-}$. Apply Stokes' Theorem and the orientation-swapping identity for line-integrals:*

$$\iint_{S_{R-}} (\nabla \times \vec{F}) \bullet d\vec{S} = \oint_{\partial S_{R-}} \vec{F} \bullet d\vec{r} = -\oint_{\partial S_{R+}} \vec{F} \bullet d\vec{r} = \boxed{-2\pi R^2.}$$



Example 7.7.5. *Once more think about the vector field $\vec{F} = \langle -y, x, z \rangle$. Notice that $\vec{F}$ is differentiable on $\mathbb{R}^3$. We can apply Stokes' Theorem to any simply connected surface. If the consistently-oriented boundary of that surface is ∂S_{R+} then the flux of $\nabla \times \vec{F}$ is $2\pi R^2$.*

Stokes' Theorem allows us to deform the flux integral of $\nabla \times \vec{F}$ over a family of surfaces which share a common boundary. What about a closed surface? A sphere, ellipsoid, or the faces comprising a cube are all examples of closed surfaces. If S is an closed surface then $\partial S = \emptyset$. Does Stokes' Theorem hold in this case?

Example 7.7.6. *Suppose S is a simply connected closed surface S and suppose $\vec{F}$ is differentiable on some open set containing S then I claim that*

S closed $\quad \partial S = \phi$

$$\iint_S (\nabla \times \vec{F}) \bullet d\vec{S} = 0.$$

To see why this is true simply cut S in halves S_1 and S_2. Notice that to consistently oriented S_1 and S_2 we must have that $\partial S_1 = -\partial S_2$. Apply Stokes' Theorem to each half to obtain:

$$\oint_{\partial S_1} \vec{F} \bullet d\vec{r} = \iint_{S_1} (\nabla \times \vec{F}) \bullet d\vec{S} \quad \& \quad \oint_{\partial S_2} \vec{F} \bullet d\vec{r} = \iint_{S_2} (\nabla \times \vec{F}) \bullet d\vec{S} \quad (\star).$$

$\partial S_1 = -\partial S_2$

Note that,

$$\iint_{S_1} (\nabla \times \vec{F}) \bullet d\vec{S} + \iint_{S_2} (\nabla \times \vec{F}) \bullet d\vec{S} = \iint_S (\nabla \times \vec{F}) \bullet d\vec{S}$$

But, $\oint_{\partial S_2} \vec{F} \bullet d\vec{r} = -\oint_{\partial S_1} \vec{F} \bullet d\vec{r}$. Therefore, addiing the eq. in $\star$ yields that $\iint_S (\nabla \times \vec{F}) \bullet d\vec{S} = 0$.

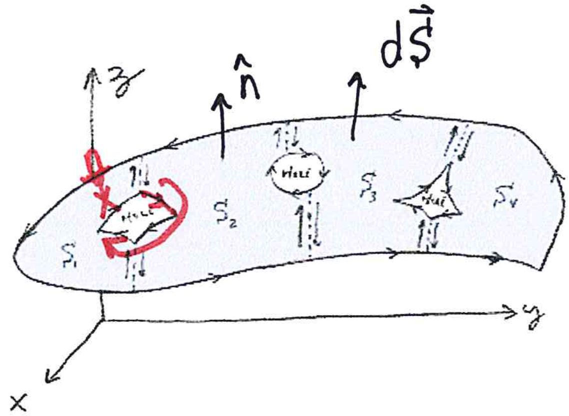
We can easily include the result of the example above by defining the integral over the empty set to be zero. Another interesting extension of the basic version of the theorem is the case that the surface has a few holes. The justification for this theorem will be a simple extension of the argument we already offered in the case of Green's Theorem.

Theorem 7.7.7. *Stokes' Theorem for connected surface possibly including holes.*

Suppose S is a connected surface S with consistently oriented boundary ∂S. If S has holes then we insist that the boundaries of the holes be oriented such that S is toward the left of the curve as we travel along the edge of the hole. Likewise, the outer boundary curve must also be oriented such that the surface if on the left as we traverse the boundary in the direction of its orientation. Suppose $\vec{F}$ is differentiable on some open set containing S then

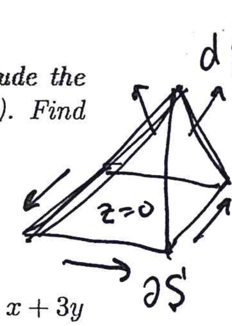
$$\oint_{\partial S} \vec{F} \bullet d\vec{r} = \iint_S (\nabla \times \vec{F}) \bullet d\vec{S}.$$

Proof: Mostly by a picture. If S is connected with holes then we can cut S into pieces which are simply connected. We then apply Stokes' Theorem to each simply connected component surface. Finally, **sum** these equations together to obtain the surface integral of the flux over the whole surface and the line-integral around the boundary.



The key feature revealed by the picture is that all the interior cuts will cancel in this **sum** since any edge which is shared by two simply connected components must be oppositely oriented when viewed as the consistent boundary of the simply connected components. $\square$.

Example 7.7.8. *Suppose S is a pyramid with square-base on the xy-plane (do not include the square-base in the surface S so the boundary of S is the square at the base of the pyramid). Find the flux of $\nabla \times \vec{F}$ through the pyramid if $\vec{F}(x,y,z) = \langle 1,3,z^3 \rangle$. Apply Stokes' Theorem,*



$$\iint_S \nabla \times \vec{F} \bullet d\vec{S} = \oint_{\partial S} \vec{F} \bullet d\vec{r}.$$

Note that $\vec{F}|_{\partial S} = \langle 1,3,0 \rangle$ since $z = 0$ on the boundary of the pyramid. Define $g(x,y,z) = x + 3y$ and note that $\nabla g = \langle 1,3,0 \rangle$ thus $\vec{F}|_{\partial S} = \langle 1,3,0 \rangle$ is conservative on the xy-plane and it follows that the integral around the closed square loop ∂S is zero. Thus,

$\vec{F}(x,y,0) = \langle 1,3,0 \rangle$

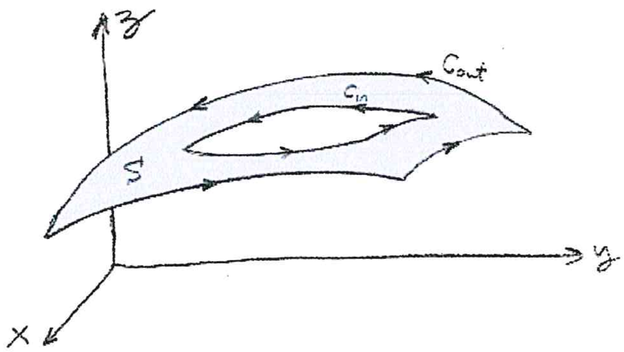
$$\iint_S (\nabla \times \vec{F}) \bullet d\vec{S} = 0.$$

This is not terribly surprising since direct computation easily shows that $\nabla \times \vec{F} = 0$.

Suppose S is a connected surface which has outer boundary C_{out} and inner boundary C_{in} where we have consistently oriented C_{out} but oppositely oriented C_{in}; $\partial S = C_{out} \cup (-C_{in})$. Applying Stokes' Theorem with holes to a vector field $\vec{F}$ which is differentiable on an open set containing S[20],

$$\oint_{C_{out}} \vec{F} \bullet d\vec{r} - \oint_{C_{in}} \vec{F} \bullet d\vec{r} = \iint_S (\nabla \times \vec{F}) \bullet d\vec{S}$$

This is a very interesting formula in the case $\nabla \times \vec{F} = 0$ on some connected annular surface:



Theorem 7.7.9. *Deformation Theorem for connected surfaces*

> Suppose S is connected with inner boundary C_{in} (oriented such that the surface's normal side is to the right of C_{in}) and outer boundary C_{out} oriented such that S is on the left of the curve then if $\vec{F}$ is differentiable on an open set containing S and has $\nabla \times \vec{F} = 0$ on S,
>
> $$\oint_{C_{out}} \vec{F} \bullet d\vec{r} = \oint_{C_{in}} \vec{F} \bullet d\vec{r}.$$

Proof: follows from the formula above the theorem $\square$.

Application to conservative vector fields: How does Stokes' Theorem help us understand conservative vector fields in $\mathbb{R}^3$? Recall we have a list of equivalent characterizations for a simply connected space U as given in Proposition 7.4.5: Suppose U is an open connected subset of $\mathbb{R}^n$ then the following are equivalent

1. $\vec{F}$ is conservative; $\vec{F} = \nabla f$ on all of U
2. $\vec{F}$ is path-independent on U
3. $\oint_C \vec{F} \bullet d\vec{r} = 0$ for all closed curves C in U
4. (add precondition $n = 3$ and U be simply connected) $\nabla \times \vec{F} = 0$ on U.

We argued before how $[\nabla \times \langle P, Q, 0\rangle] \bullet \hat{z} = \partial_x Q - \partial_y P = 0$ paired with the deformation version of Green's Theorem allowed us to shrink loop integrals to a point hence estabishing that $\vec{F} = \langle P, Q, 0\rangle$

[20] if S were a donut this does not necessitate that $\vec{F}$ be differentiable in the center of the big-circle of the donut, it merely means $\vec{F}$ is differentiable near where the actual donut is found

is conservative if it passed Clairaut's test ($\partial_y P = \partial_x Q$ on a simple connected two-dimensional domain. Let us continue to three dimensions now that we have the needed technology.

Suppose $\vec{F} = \langle P, Q, R \rangle$ has vanishing curl ($\nabla \times \vec{F} = 0$) on some simply connected subset U of $\mathbb{R}^3$. Suppose C_1 and C_2 are any two paths from P to Q in U. Observe that $C_1 \cup (-C_2)$ bounds a simply connected surface S on which $\nabla \times \vec{F} = 0$ (since S sits inside U, and U has no holes). Apply Stokes' Theorem to S:

$$\oint_{C_1 \cup (-C_2)} \vec{F} \bullet d\vec{r} = \iint_S (\nabla \times \vec{F}) \bullet d\vec{S} = 0$$

Therefore, $\int_{C_1} \vec{F} \bullet d\vec{r} = \int_{C_2} \vec{F} \bullet d\vec{r}$ and path-independence on U follows.

Discussion: criteria 1,2 and 3 are n-dimensional results since the arguments we gave apply equally well in higher dimensions. However, item 4 is only worthile in it's application to $n = 3$ or with proper specialization $n = 2$ because we have no cross-product and hence no curl in dimension $n = 4, 5...$ etc.... You might wonder what is the generalization of (4.) for vector fields in $\mathbb{R}^n$. The answer involves differential forms. The exterior derivative allows us to properly extend vector differentiation to n-dimensions. This is not just an *academic*[21] comment, in the study of differential equations we enjoy solving exact differential equation. If $Pdx + Qdy + Rdz = 0$ then we can solve by $f(x, y, z) = 0$ if we can find f with $\partial_x f = P$, $\partial_y f = Q$ and $\partial_z f = R$. But, this problem is one we have already solved:

$$Pdx + Qdy + Rdz = 0 \text{ is exact} \qquad \leftrightarrows \qquad \vec{F} = \langle P, Q, R \rangle = \nabla f$$

Thus $\nabla \times \langle P, Q, R \rangle = 0$ on simply connected $U \subset \mathbb{R}^3$ implies existence of solutions for the given differential equation $Pdx+Qdy+Rdz = 0$. What about the case of additional indpendent variables; suppose w, x, y, z are variables

$$Idw + Pdx + Qdy + Rdz = 0 \text{ is exact} \qquad \leftrightarrows \qquad \vec{F} = \langle I, P, Q, R \rangle = \nabla f$$

If we could find the condition analogue to $\nabla \times \vec{F} = 0$ then we would have a nice test for the existence of solutions to the given DEqn. It turns out that the test is simply given by the exterior derivative of the given DEqn; [22] If $P_1 dx_1 + P_2 dx_2 + \cdots + P_n dx_n = 0$ then this DEQn is exact on a simply connected domain in $\mathbb{R}^n$ iff $d[P_1 dx_1 + P_2 dx_2 + \cdots + P_n dx_n] = 0$. Consult my advanced calculus notes and/or ask me for details about what "d" means in the context above.

Naturally, we can extend the annular result to the more general case that the surface as finitely-many holes:

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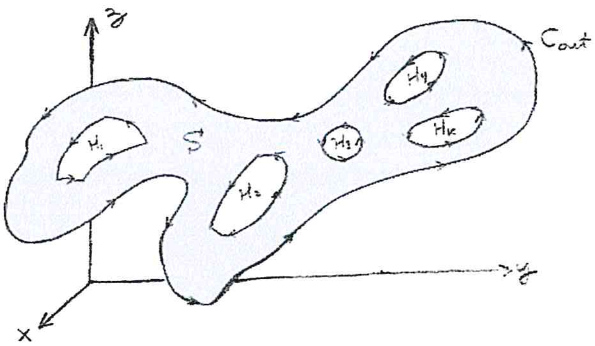
[21] sad comment on our culture that this is an insult!

[22] example: $x^2dy - ydx = 0$ is not exact since $d[x^2dy - ydx] = 2xdx \wedge dy - dy \wedge dx \neq 0$. In contrast $ydx + xdy = 0$ is exact since $d[ydx + xdy] = dy \wedge dx + dx \wedge dy = 0$. We discussed the wedge product and exterior derivative in lecture, *ask if interested and missed it...*



Theorem 7.7.10. *Stokes' Theorem for connected surface possibly including holes.*

Suppose S is a connected surface S with consistently oriented boundary ∂S. If S has k holes $H_1, H_2, \dots, H_k$ then $\partial S = C_{out} \cup (-\partial H_1) \cup (-\partial H_2) \cdots (-\partial H_k)$. Suppose $\vec{F}$ is differentiable on some open set containing S then

$$\oint_{C_{out}} \vec{F} \bullet d\vec{r} - \oint_{\partial H_1} \vec{F} \bullet d\vec{r} - \oint_{\partial H_2} \vec{F} \bullet d\vec{r} - \cdots - \oint_{\partial H_k} \vec{F} \bullet d\vec{r} = \iint_S (\nabla \times \vec{F}) \bullet d\vec{S}.$$

Remark 7.7.11.

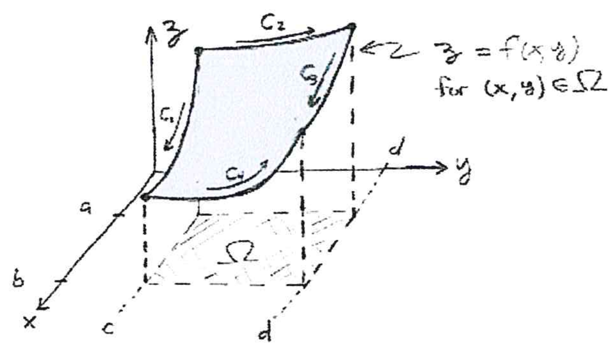
Electrostatics for a curved surface is an interesting problem. Imagine that the electric field was confined to the surface. We considered this problem in some depth for the plane. It is worth mentioning that we could just as well repeat those arguments here if we wished to model some field which is bound to flow along a surface. The details of the theory would depend on the particulars of the surface. I leave this as an open problem for the interested reader. You might even think about what surface you could pick to force the field to have certain properties... this is a prototype for the idea used in string theory; the geometry of the underlying space derives the physics. At least, this is one goal, sometimes realized...

7.7.1 proof of Stokes' theorem for a graph

Our goal is to show that $\oint_{\partial S} \vec{F} \bullet d\vec{r} = \iint_S (\nabla \times \vec{F}) \bullet d\vec{S}$ for a simply connected surface S which can be expressed as a graph. Suppose $z = f(x,y)$ for $(x,y) \in \Omega$ In particular, as a starting point, let $\Omega = [a,b] \times [c,d]$[23]. It is easily calculated that $\vec{r}(x,y) = \langle x, y, f(x,y) \rangle$ induces normal vector field $\vec{N}(x,y) = \langle -\partial_x f, -\partial_y f, 1 \rangle$. The boundary of S consists of 4 line segments: $C_1 \cup C_2 \cup (-C_3) \cup (-C_4)$ where

1. C_1: for $a \leq t \leq b$ we set $x = t,\ y = c,\ z = f(t,c)$ hence $dx = dt,\ dy = 0,\ dz = \partial_x f(t,c)dt$
2. C_2: for $c \leq t \leq d$ we set $x = a,\ y = t,\ z = f(a,t)$ hence $dx = 0,\ dy = dt,\ dz = \partial_y f(a,t)dt$
3. C_3: for $a \leq t \leq b$ we set $x = t,\ y = d,\ z = f(t,d)$ hence $dx = dt,\ dy = 0,\ dz = \partial_x f(t,d)dt$
4. C_4: for $c \leq t \leq d$ we set $x = b,\ y = t,\ z = f(b,t)$ hence $dx = 0,\ dy = dt,\ dz = \partial_y f(b,t)dt$.

We could visualize it as follows:



Consider a vector field $\vec{F} = \langle P, Q, R \rangle$ which is differentiable on some open set containing S. Calculate, for reference in the calculations below,

$$\nabla \times \vec{F} = \langle\ \partial_y R - \partial_z Q,\ \partial_z P - \partial_x R,\ \partial_x Q - \partial_y P\ \rangle$$

To calculate the flux of $\nabla \times \vec{F}$ we need to carefully compute $(\nabla \times \vec{F}) \bullet \vec{N}$;

$$\begin{aligned}(\nabla \times \vec{F}) \bullet \vec{N} &= \langle\ \partial_y R - \partial_z Q,\ \partial_z P - \partial_x R,\ \partial_x Q - \partial_y P\ \rangle \bullet \langle -\partial_x f, -\partial_y f, 1 \rangle \\ &= \big[\partial_z Q \partial_x f + \partial_x Q\big] - \big[\partial_z P \partial_y f + \partial_y P\big] + \big[\partial_x R \partial_y f - \partial_y R \partial_x f\big]\end{aligned}$$

To proceed we break the problem into three. In particular $\vec{F} = \vec{F_1} + \vec{F_2} + \vec{F_3}$ where we let $\vec{F_1} = \langle P, 0, 0 \rangle$, $\vec{F_2} = \langle 0, Q, 0 \rangle$ and $\vec{F_3} = \langle 0, 0, R \rangle$. For $\vec{F_1} = \langle P, 0, 0 \rangle$ we calculate:

$$\begin{aligned}\iint_S (\nabla \times \vec{F_1}) \bullet d\vec{S} &= \int_a^b \int_c^d (-\partial_z P \partial_y f - \partial_y P) dy\, dx \\ &= -\int_a^b \int_c^d \partial_y \big[P(x, y, f(x,y))\big] dy\, dx \qquad \text{(chain-rule)} \\ &= \int_a^b \big[P(x, c, f(x,c)) - P(x, d, f(x,d))\big] dx. \quad (\star)\end{aligned}$$

[23]In my first attempt I tried Ω as a type-I region given by functions f_1, f_2 such that $\Omega = \{(x,y) \mid f_1(x) \leq y \leq f_2(x),\ a \leq x \leq b\}$, however, this is too technical, it is clearer to show how this works for a rectangular domain.

On the other hand, we calculate the circulation of $\vec{F}_1 = P\,\widehat{x}$ around ∂S note that $dx = 0$ for C_2 and C_4 hence only C_1 and C_3 give nontrivial results,

$$\begin{aligned}
\int_{\partial S} \vec{F}_1 \bullet d\vec{r} = \int_{\partial S} Pdx &= \int_{C_1} Pdx - \int_{C_3} Pdx \\
&= \int_a^b P(t,c,f(t,c))dt - \int_a^b P(t,d,f(t,d))dt \\
&= \int_a^b \big[P(x,c,f(x,c)) - P(x,d,f(x,d))\big]dx.
\end{aligned}$$

Consquently we have established Stokes' Theorem for $\vec{F}_1$ over our rather simple choice of surface. Continuing, consider $\vec{F}_2 = Q\,\widehat{y}$. Calculate, given our experience with the Pdx integrals we need not meet in the middle this time, I offer a direct computation:

$$\begin{aligned}
\iint_S (\nabla \times \vec{F}_2) \bullet d\vec{S} &= \int_a^b \int_c^d \big(\partial_z Q \partial_x f - \partial_x Q\big)\, dy\, dx \\
&= \int_c^d \int_a^b \partial_x \big[Q(x,y,f(x,y))\big]\, dy\, dx \qquad \text{(chain-rule, \& swapped bounds)} \\
&= \int_c^d Q(b,y,f(b,y))dy - \int_c^d Q(a,y,f(a,y))dy \\
&= \int_c^d Q(b,t,f(b,t))dt - \int_c^d Q(a,t,f(a,t))dt \\
&= \int_{C_2} Qdy - \int_{-C_4} Qdy \\
&= \oint_{\partial S} Qdy \qquad \text{(integrals along } C_1 \text{ and } C_3 \text{ are zero)} \\
&= \oint_{\partial S} \vec{F}_2 \bullet d\vec{r}.
\end{aligned}$$

Next, we work out Stokes' Theorem for $\vec{F}_3 = R\,\widehat{z}$. I'll begin with the circulation this time,

$$\begin{aligned}
\oint_{\partial S} \vec{F}_3 \cdot d\vec{r} &= \int_{C_1} R dz - \int_{C_3} R dz + \int_{C_2} R dz - \int_{C_4} R dz \\
&= \int_a^b R(t,c,f(t,c))\frac{\partial f}{\partial x}(t,c)\;dt - \int_a^b R(t,d,f(t,d))\frac{\partial f}{\partial x}(t,d)\;dt \\
&\quad + \int_c^d R(b,t,f(b,t))\frac{\partial f}{\partial y}(b,t)dt - \int_c^d R(a,t,f(a,t))\frac{\partial f}{\partial y}(a,t)dt \\
&= -\int_a^b \left(R(x,d,f(x,d))\frac{\partial f}{\partial x}(x,d) - R(x,c,f(x,c))\frac{\partial f}{\partial x}(x,c) \right) dx \\
&\quad + \int_c^d \left(R(b,y,f(b,y))\frac{\partial f}{\partial y}(b,y) - R(a,y,f(a,y))\frac{\partial f}{\partial y}(a,y) \right) dy \\
&= -\int_a^b \left[R(x,y,f(x,y))\frac{\partial f}{\partial x}(x,y) \right]\Bigg|_c^d dx + \int_c^d \left[R(x,y,f(x,y))\frac{\partial f}{\partial y}(x,y) \right]\Bigg|_a^b dy \\
&= -\int_a^b \int_c^d \frac{\partial R}{\partial y}\frac{\partial f}{\partial x}\,dy\;dx + \int_c^d \int_a^b \frac{\partial R}{\partial x}\frac{\partial f}{\partial y}\,dx\;dy \\
&= \int_a^b \int_c^d \left(\frac{\partial R}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial R}{\partial y}\frac{\partial f}{\partial x} \right) dy\,dx \\
&= \iint_S (\nabla \times \vec{F}_3) \cdot d\vec{S}.
\end{aligned}$$

Therefore, by linearity of the curl and line and surface integrals we find that

$$\oint_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}.$$

Notice that the choice of rectangular bounds for Ω allowed us to freely exchange the order of integration since x and y bounds were independent. If Ω was a less trivial type I or type II region, then the arguments given in this section need some modification since swapping bounds is in general a somewhat involved process. That said, the result just proved is quite robust when paired with the earlier polyhedral proof to make a general argument. If surface consists of a graph with a curved domain then we can break it into rectangular subdomains and apply the result of this section to each piece. Once more when we sum those results together the nature of the adjoining regions is to cancel all line integrals modulo the boundary of the overall surface.[24] If the surface does not admit presentation as a graph $z = f(x,y)$ then generally we can patch it together with several graphs[25]. We apply the result of this section to each such patch and the sum the results to obtain Stokes' Theorem for a general simply connected surface.

[24] Technically, we'll have to form a sequence of such regions for some graphs and then take the limit as the rectangular net goes to infinitely many sub-divisions, however, the details of such analysis are beyond the scope of these notes. If this seems similar to the proof we presented for Green's Theorem then your intuition may serve you well in the remainder of this course.

[25] could be $z = f(x,y)$ type, or $y = g(x,z)$ or $x = h(y,z)$, the implicit function theorem of advanced calcululus will give a general answer to how this is done for a level surface