LECTURE 37: GAUSS' Th^{m} AKA THE DIVERGENCE Th^{m}

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$$\iiint_B (\nabla \cdot \vec{F})\, dV = \iint_{\partial B} \vec{F} \cdot d\vec{S}$$

7.8 gauss theorem aka Divergence.

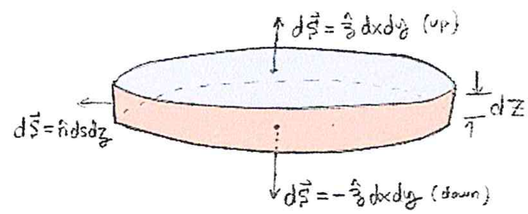
Warning: the argument that follows is an infinitesimal argument. To properly undestand the meaning of this discussion we should remember this is simply notation for a finite approximation where a certain limit is taken. That said, I will not clutter this argument with limits. I leave those to the reader here. Also... I will offer another less heuristic argument towards the end of this section, I prefer this one since it connects with our previous discussion about the divergence of a two-dimensional vector field and Green's Theorem.

Green's Theorem in the plane quantifies the divergence of the vector field $P\,\widehat{x} + Q\,\widehat{y}$ through the curve ∂D;

$$\int_{\partial D} (P\,\widehat{x} + Q\,\widehat{y}) \bullet \widehat{n}\, ds = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dx\, dy$$

Suppose we consider a three-dimensional vector field $\vec{F} = P\,\widehat{x} + Q\,\widehat{y} + R\,\widehat{z}$. Furthermore, suppose we consider an infinitesimal cylinder $E = D \times dz$.

$$E = D \times \underbrace{[z_0, z_0+\Delta z]}_{I}$$



$$\Delta z = \int_I dz = z \Big|_{z_0}^{z_0+\Delta z} = \Delta z$$

What is the flux of $\vec{F}$ out of the cylinder? Apply Green's Theorem to see that the flux through the vertical faces of the cylinder are simply given by either of the expressions below:

$$\Phi_{hor} = \iint_{\partial D \times [z, z+dz]} (P\,\widehat{x} + Q\,\widehat{y}) \bullet \widehat{n}\, ds\, \overset{\Delta z}{dz} = \iiint_{D \times [z,z+dz]} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dx\, dy\, dz$$

Since $d\vec{S} = \widehat{n} ds\, dz$ is clearly the vector area element of the vertical face(s) of the cylinder it is clear that the double integral above is simply the surface integral over the vertical faces of the cylinder. We identify,

$$\Phi_{hor} = \iint_{\partial D \times [z,z+dz]} \vec{F} \bullet d\vec{S} = \iiint_{D \times [z,z+dz]} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dx\, dy\, dz.$$

On the other hand, the flux through the horizontal caps of the cylinder $D \times \{z\}$ and $D \times \{z + dz\}$ involve only the z-component of $\vec{F}$ since $d\vec{S} = \widehat{z} dx\, dy$ for the upper cap and $d\vec{S} = -\,\widehat{z} dx\, dy$ for the lower cap hence the fluxes are

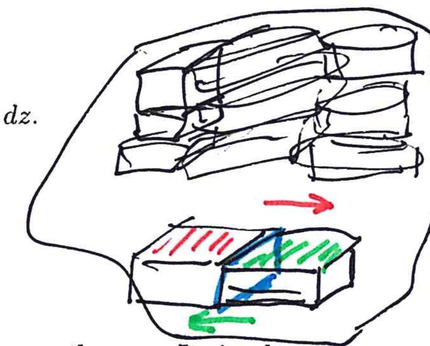
$$\Phi_{up} = \iint_D R(x, y, z + dz)\, dx\, dy \qquad \& \qquad \Phi_{down} = \iint_D -R(x, y, z)\, dx\, dy$$

The sum of these gives the net vertical flux:

$$\Phi_{vert} = \iint_D (\ R(x, y, z + dz) - R(x, y, z)\ )\, dx\, dy = \iiint_{D \times [z,z+dz]} \frac{\partial R}{\partial z} dx\, dy\, dz.$$

where in the last step we used the FTC to rewrite the difference as an integral. To summarize,

$$\Phi_{vert} = \iint_{caps} \vec{F} \bullet d\vec{S} = \iiint_{D\times[z,z+dz]} \frac{\partial R}{\partial z} dx\,dy\,dz.$$



The net-flux through the cylinder is the sum $\Phi_{vert} + \Phi_{hor}$. We find that,

$$\iint_{\partial E} \langle P, Q, R\rangle \bullet d\vec{S} = \iiint_E \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\,dy\,dz.$$

I prefer to write the result as follows:

$$\boxed{\iint_{\partial E} \vec{F} \bullet d\vec{S} = \iiint_E \nabla \bullet \vec{F}\, dV.}$$

This is the celebrated theorem of Gauss. We often refer to it as the *divergence theorem*. It simply says that the net flux through a surface is portional to the continuous sum of the divergence throughout the solid. In other words, the divergence of a vector field $\vec{F}$ measures the number of field lines flowing from a particular volume. We found the two-dimensional analogue of this in our analysis of Green's Theorem and this is the natural three-dimensional extension of that discussion. For future reference: (this is also called Gauss' Theorem)

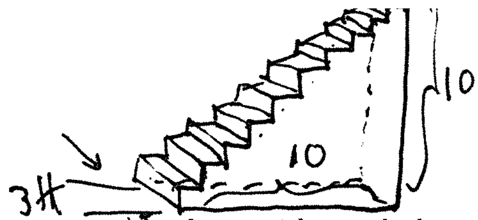
Theorem 7.8.1. *divergence theorem for a simple solid*

Suppose E a simple solid (has no interior holes) with consistently oriented outward facing boundary ∂E. If $\vec{F}$ is differentiable on an open set containing E then,

$$\iint_{\partial E} \vec{F} \bullet d\vec{S} = \iiint_E \nabla \bullet \vec{F}\, dV.$$

Discussion: But, I only proved it for a cylinder? Is this needed? Does it apply to other shapes? Yes. Consider the case that D is a rectangular region. We can use the argument offered above to obtain Gauss' theorem for a rectangular solid. Take any other simple solid (one with no holes) and note that you can obtain the solid as a union (possibly infinite!) of rectangular solids. Positively orient each rectangular solid and apply Gauss' Theorem to each member of the partition. Next, add the results together. On the one side we obtain the volume integral of the divergence. On the other side we get a sum of flux over many rectangular solids, some with adjacent faces. Think about this, any interior face of a particular rectangular solid will share a face with another member of the partition. Moreover, the common faces must be oppositely oriented in the distinct, but adjacent, rectangular solids. Thus, the interior flux all cancels leaving only the outside faces. The sum of the flux over all outside faces is simply the surface integral over the boundary of the simple solid. In this way we extend Gauss' Theorem to any solid without holes. Naturally, this leaves something to be desired analytically, but you can also appreciate this argument is very much the same we gave for Green's Theorem. This would seem to be part of some larger story...but, that is a story for another day[26].

[26] look-up a proof of the generalized Stokes' Theorem in an advanced calculus text if you are interested. The key construction involves generalizing the polyhedral decomposition to something called an n-chain or perhaps an n-simplex depending what you read. Basically, you need some way of breaking the space into oriented parts with nice oriented boundaries, you prove the theorem for one such item and extrapolate via face-cancelling arguments as we have seen here in this case



Example 7.8.2. *Suppose $\vec{F}(x,y,z) = \langle x+y, y+x, z+y\rangle$ and you wish to calculate the flux of $\vec{F}$ through a set of stairs which has width 3 and 10 steps which are each height and depth 1. Let E be the set of stairs and ∂E the outward-oriented surface. Clearly the calculation of the flux over the surface of the stairs would be a lengthy and tedious computation. However, note that $\nabla \cdot \vec{F} = \frac{\partial(x+y)}{\partial x} + \frac{\partial(y+x)}{\partial y} + \frac{\partial(z+y)}{\partial z} = 3$ hence we find by Gauss' Theorem:*

$$\iint_{\partial E} \vec{F} \cdot d\vec{S} = \iiint_E \nabla \cdot \vec{F}\, dV = 3\iiint_E dV = 3Vol(E).$$

Elementary school math shows:

$$Vol(E) = 3(1+2+3+4+5+6+7+8+9+10) = 165$$

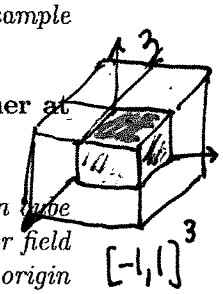
Hence, $\boxed{\iint_{\partial E} \vec{F} \cdot d\vec{S} = 495.}$

Challenge: work the previous example for n-steps.

Example 7.8.3. *Suppose $\vec{F}$ is a differentiable at all points near a simple solid E. Calculate the flux of the curl of $\vec{F}$ through ∂E:*

$$\iint_{\partial E} (\nabla \times \vec{F}) \cdot d\vec{S} = \iiint_E \nabla \cdot (\nabla \times \vec{F}) dV = \iiint_E (0) dV = 0.$$

I used the identity $\nabla \cdot (\nabla \times \vec{F}) = 0$. You can contrast this argument with the one given in Example 7.7.6. Both examples are worth study.



Example 7.8.4. Problem: Consider the cube E with side-length L and one corner at the origin. Calculate the flux of $\vec{F} = \langle x, y, z\rangle$ through the upper face of the cube.

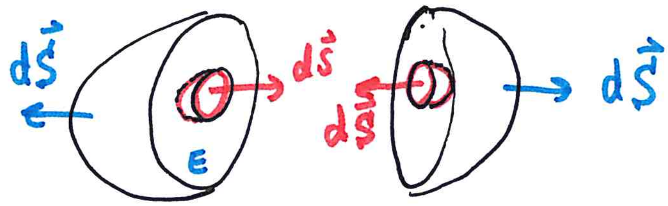
Solution: *Note that we cannot use a simple symmetry argument to see it is 1/6 of the given cube since the face in question differs from the base face (for example) in its relation to the vector field $\vec{F}$. On the other hand, if we imagine a larger cube of side-length $2L$ which is centered at the origin then the vector field is symmetric with respect to the faces of $[-L,L]^3$. Call this larger cube E' and observe that we can easily calculate the net-flux through $\partial E'$ by the divergence theorem.*

$$\Phi_{\partial E'} = \iint \vec{F} \cdot d\vec{S} = \iiint_{E'} \nabla \cdot \vec{F} dV = \iiint_{E'} 3\, dV = 3Vol(E') = 3(2L)^3 = 24L^3.$$

Notice that the face $[0,L] \times [0,L] \times \{L\}$ is 1/4 of the upper face of E' and it is symmetric with respect to the other 3/4 of the face $[-L,L] \times [-L,L] \times \{L\}$ with regard to $\vec{F}$. It follows the flux through the upper face of E is 1/4 of the flux through the upper face of E'. Moreover, since the faces of E' are symmetric with regard to $\vec{F}$ we find that 1/6 of the total flux through $\partial E'$ passes through that upper face of E'. In summary, the flux through the face in question is simply 1/24 of the total flux through $\partial E'$ and the flux through the upper face of E is $\boxed{L^3.}$

It should come as no suprise that there is a simple argument to extend the divergence theorem to a solid with a hole(s) in it. Suppose E is a solid which has a hole H in it. Denote the boundary of E by $\partial E = S_{out} \cup S_{in}$ where these surfaces are oriented to point out of E. Notice we can do surgery on E and slice it in half so that the remaining parts are simple solids (with no holes). The picture below illustrates this basic cut.

should include picture here



More exotic holes require more cutting, but the idea remains the same, we can cut any solid with a finite number of holes into a finite number of simple solids. Apply the divergence theorem, for an appropriately differentiable vector field, to each piece. Then add these together, note that the adjacent face's flux cancel leaving us the simple theorem below:

Theorem 7.8.5. *divergence theorem for a solid with k-interior holes.*

Suppose E a solid with interior holes $H_1, H_2, \dots, H_k$. Orient the surfaces $S_1, S_2, \dots, S_k$ of the holes such that the normals point into the holes and orient the outer surface S_{out} of E to point outward; hence $\partial E = S_{out} \cup S_1 \cup S_2 \cup \cdots \cup S_k$ gives E an outward oriented boundary. If $\vec{F}$ is differentiable on an open set containing E then,

$$\iint_{\partial E} \vec{F} \bullet d\vec{S} = \iiint_E \nabla \bullet \vec{F}\, dV.$$

This is perhaps more interesting if we take the holes as solids on their own right with outward oriented surfaces $\partial H_1, \partial H_2, \dots, \partial H_k$ (this makes $\partial H_i = -S_i$ for $i = 1, 2, \dots, k$. It follows that:

$$\iint_{\partial S_{out}} \vec{F} \bullet d\vec{S} - \iint_{\partial H_1} \vec{F} \bullet d\vec{S} - \iint_{\partial H_2} \vec{F} \bullet d\vec{S} - \cdots - \iint_{\partial H_k} \vec{F} \bullet d\vec{S} = \iiint_E \nabla \bullet \vec{F}\, dV.$$

Note that we have an interesting result if $\nabla \bullet \vec{F} = 0$ on the E described above. In that case we obtain a deformation theorem for the flux spreading between surfaces:

Theorem 7.8.6. *deformation theorem for a solid with k-interior holes.*

Suppose E a solid with interior holes $H_1, H_2, \dots, H_k$. Let the outer surface S_{out} of E be oriented to point outward and give the hole surfaces an orientation which points out of the hole; If $\vec{F}$ is differentiable on an open set containing E and $\nabla \bullet \vec{F} = 0$ then,

$$\iint_{\partial S_{out}} \vec{F} \bullet d\vec{S} = \iint_{\partial H_1} \vec{F} \bullet d\vec{S} + \iint_{\partial H_2} \vec{F} \bullet d\vec{S} + \cdots + \iint_{\partial H_k} \vec{F} \bullet d\vec{S}.$$

This theorem forms the basis for the three-dimensional electrostatics and much more. Basically it says that if a field is mostly divergence free then the flux comes from only those places where the divergence is non-vanishing. Everywhere else the field lines just spread out. Given $\vec{F}$ with $\nabla \bullet \vec{F} = 0$ most places, I think of the holes as where the charge for the field is, either **sinks** or **sources**. From these mysterious holes the field lines flow in and out.

7.8.1 three-dimensional electrostatics

The fundamental equation of electrostatics is known as *Gauss' Law.* In three dimensions it simply states that the flux through a closed surface is proportional to the charge which is enclosed.

$$\Phi_E = Q_{enc}$$

In particular, if we denote $\sigma = dQ/dV$ and have in mind the solid E with boundary ∂E,

$$\oint_{\partial E} (\vec{E} \bullet d\vec{S} = \iiint_E \sigma dV$$

Suppose we have an isolated charge Q at the origin and we apply Gauss law to a sphere of radius ρ centered at the origin then we can argue by symmetry the electric field must be entirely radial in direction and have a magnitude which depends only on ρ. It follows that:

$$\oint_{\partial E} (\vec{E} \bullet d\vec{S} = \iiint_E \delta dV \quad \Rightarrow \quad (4\pi\rho^2)E = Q$$

Hence, the **coulomb field** in three dimensions is as follows:

$$\boxed{\vec{E}(\rho, \phi, \theta) = \frac{Q}{4\pi\rho^2}\widehat{\rho}}$$

Let us calculate the flux of the Coulomb field through a sphere S_R of radius R:

$$\begin{aligned}
\oint_{S_R} (\vec{E} \bullet d\vec{S} &= \int_{S_R} \bigg(\frac{Q}{4\pi\rho^2}\widehat{\rho} \bullet \widehat{\rho} dS \\
&= \int_{S_R} \frac{Q}{4\pi R^2} dS \\
&= \frac{Q}{4\pi R^2} \int_{S_R} dS \\
&= \frac{Q}{4\pi R^2}(4\pi R^2) \\
&= Q.
\end{aligned} \tag{7.3}$$

The sphere is complete. In other words, the Coulomb field derived from Gauss' Law does in fact satisfy Gauss Law in the plane. This is good news. Let's examine the divergence of this field. It appears to point away from the origin and as you get very close to the origin the magnitude of E is unbounded. It will be convenient to reframe this formula for the Coulomb field by

$$\vec{E}(x, y, z) = \frac{Q}{4\pi(x^2 + y^2 + z^2)^{3/2}}\langle x, y, z\rangle = \frac{Q}{4\pi\rho^3}\langle x, y, z\rangle.$$

Note that as $\rho = \sqrt{x^2+y^2+z^2}$ it follows that $\partial_x \rho = x/\rho$ and $\partial_y \rho = y/\rho$ and $\partial_z \rho = z/\rho$. Consequently:

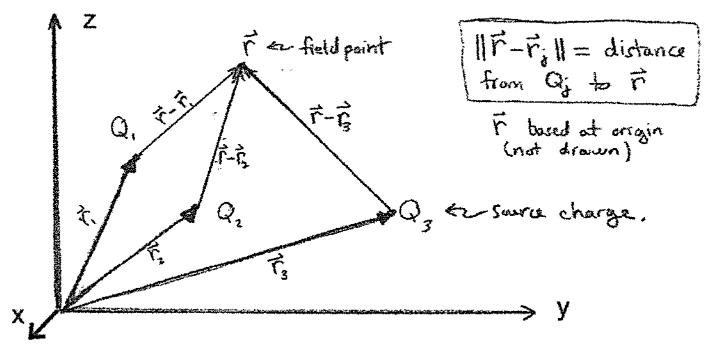
$$\begin{aligned}
\nabla \bullet \vec{E} &= \frac{Q}{4\pi}\left(\frac{\partial}{\partial x}\left[\frac{x}{\rho^3}\right] + \frac{\partial}{\partial y}\left[\frac{y}{\rho^3}\right] + \frac{\partial}{\partial z}\left[\frac{z}{\rho^3}\right]\right) \\
&= \frac{Q}{4\pi}\left(\frac{\rho^3 - 3x\rho^2\partial_x\rho}{\rho^6} + \frac{\rho^3 - 3y\rho^2\partial_y\rho}{\rho^6} + \frac{\rho^3 - 3z\rho^2\partial_z\rho}{\rho^6}\right) \\
&= \frac{Q}{4\pi}\left(\frac{\rho^3 - 3x^2\rho}{\rho^6} + \frac{\rho^3 - 3y^2\rho}{\rho^6} + \frac{\rho^3 - 3z^2\rho}{\rho^6}\right) \\
&= \frac{Q}{4\pi}\left(\frac{3\rho^3 - 3\rho(x^2+y^2+z^2)}{\rho^6}\right) \\
&= 0.
\end{aligned}$$

If we were to carelessly apply the divergence theorem this could be quite unsettling: consider,

$$\iint_{\partial E} \vec{E} \bullet d\vec{S} = \iiint_E \nabla \bullet \vec{E}\, dV \quad \Rightarrow \quad Q = \iiint_E (0) dV = 0.$$

But, Q need not be zero hence there is some contradiction? Why is there no contradiction? Can you resolve this paradox?

Moving on, suppose we have N charges placed at source points $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N$ then we can find the total electric field by the principle of superposition.



We simply take the vector sum of all the coulomb fields. In particular,

$$\vec{E}(\vec{r}) = \sum_{j=1}^{N} \vec{E}_j = \sum_{j=1}^{n} \frac{Q_j}{4\pi} \frac{\vec{r} - \vec{r}_j}{||\vec{r} - \vec{r}_j||^3}$$

What is the flux through a sphere which encloses just the k-th one of these charges? Suppose S_R is a sphere of radius R centered at $\vec{r}_k$. We can calculate that

$$\iint_{S_R} \vec{E}_k \bullet d\vec{S} = Q_k$$

whereas, since $\vec{E}_j$ is differentiable inside all of S_R for $j \neq k$ and $\nabla \bullet \vec{E}_j = 0$ we can apply the divergence theorem to deduce that

$$\iint_{S_R} \vec{E}_j \bullet d\vec{S} = 0.$$

Therefore, summing these results together we derive for $\vec{E} = \vec{E}_1 + \cdots + \vec{E}_k + \cdots + \vec{E}_N$ that

$$\iint_{S_R} \vec{E} \bullet d\vec{S} = Q_k$$

Notice there was nothing particularly special about Q_k so we have derived this result for each charge in the distribution. If we take a sphere around a charge which contains just one charge then Gauss' Law applies and the flux is simply the charge enclosed. Denote $S_1, S_2, \dots, S_N$ as little spheres which enclose the charges $Q_1, Q_2, \dots, Q_N$ respective. We have

$$Q_1 = \iint_{S_1} \vec{E} \bullet d\vec{S}, \quad Q_2 = \iint_{S_2} \vec{E} \bullet d\vec{S}, \quad \dots, \quad Q_N = \iint_{S_N} \vec{E} \bullet d\vec{S}$$

Now suppose we have a surface S which encloses all N of the charges. The electric field is differentiable and has vanishing divergence at all points except the location of the charges. In fact, the superposition of the coulomb fields has vanishing divergence ($\nabla \bullet \vec{E} = 0$) everywhere except the location of the charges. It just has the isolated singularities where the charge is found. We can apply deformation theorem version of the divergence theorem to arrive at Gauss' Law for the distribution of N-charges:

$$\iint_{S} \vec{E} \bullet d\vec{S} = \iint_{S_1} \vec{E} \bullet d\vec{S} + \iint_{S_2} \vec{E} \bullet d\vec{S} + \cdots + \iint_{S_N} \vec{E} \bullet d\vec{S} = Q_1 + Q_2 + \cdots + Q_N$$

You can calculate the divergence is zero everywhere except at the location of the source charges. Moral of story: even one point thrown out of a domain can have dramatic and global consequences for the behaviour of a vector field. In physics literature you might find the formula to describe what we found by a *dirac-delta function* these distributions capture certain infinities and let you work with them. For example: for the basic coulomb field with a single point charge at the origin $\vec{E}(\rho, \phi, \theta) = \frac{Q}{4\pi\rho^2}\hat{\rho}$ this derived from a charge density function σ which is zero everywhere except at the origin. Somehow $\iiint_E \sigma dV = Q$ for any region R which contains $(0,0,0)$. Define $\sigma(\vec{r}) = Q\delta(\vec{r})$. Where we define: for any function f which is continuous near 0 and any solid region E which contains the origin

$$\int_E f(\vec{r})\delta(\vec{r})dV = f(0)$$

and if E does not contain $(0,0,0)$ then $\iint_E f(\vec{r})\delta(\vec{r})dV = 0$. The dirac delta function turns integration into evaluation. The dirac delta function is not technically a function, in some sense it is zero at all points and infinite at the origin. However, we insist it is manageably infinity in the way just described. Notice that it does at least capture the right idea for density of a point charge: suppose E contains $(0,0,0)$,

$$\iint_E \sigma dV = \iint_E Q\delta(\vec{r})dV = Q.$$

On the other hand, we can better understand the divegence calculation by the following calculations[27]:

$$\nabla \bullet \frac{\hat{\rho}}{\rho^2} = 4\pi\delta(\vec{r}).$$

Consequentlly, if $\vec{E} = \frac{Q}{4\pi}\frac{\hat{\rho}}{\rho^2}$ then $\nabla \bullet \vec{E} = \nabla \bullet \left[\frac{Q}{4\pi}\frac{\hat{\rho}}{\rho^2}\right] = \frac{Q}{4\pi}\nabla \bullet \frac{\hat{\rho}}{\rho^2} = Q\delta(\vec{r})$. Now once more apply Gauss' theorem to the Coulomb field. This time appreciate that the divergence of $\vec{E}$ is not strictly

[27] I don't intend to explain where this 4π comes from, except to tell you that it must be there in order for the extension of Gauss' theorem to work out nicely.

zero, rather, the dirac-delta function captures the divergence: recall the RHS of this calculation followed from direct calculation of the flux of the Coloumb field through the circle ∂R,

$$\iint_{\partial E} \vec{E} \cdot d\vec{S} = \iiint_E \nabla \cdot \vec{E}\, dV \quad \Rightarrow \quad Q = \iiint_E Q\delta(\vec{r}) dV = Q.$$

All is well. This is the way to extend Gauss' theorem for coulomb fields. The fields discussed in this section are the ones found in nature for the most part. Electric fields do propagate in three-dimensions and that means that isolated charges establish a coulomb field. In a later section of this chapter we seek to describe how a continuous distribution of charge can generate a field. At the base of that discussion are the ideas presented here, although, we will not have need of the dirac-delta for the continuous smeared out charge. Some physicists argue that there is no such thing as a point charge because the existence of such a charge comes with some nasty baggage. For example, if you calculate the total energy of the Coulomb field for a single point charge you find there is infinite energy in the field. Slightly unsettling.

7.8.2 proof of divergence theorem for a rectangular solid

Suppose $\vec{F}$ is differentiable near the solid $E = [x_1, x_2] \times [y_1, y_2] \times [z_1, z_2]$. Denote the faces of the solid as follows:

$$S_1 : \quad x = x_1, (z, y) \in [z_1, z_2] \times [y_1, y_2] \text{ has } d\vec{S} = -\,\widehat{x}\, dz\, dy$$

$$S_2 : \quad x = x_2, (y, z) \in [y_1, y_2] \times [z_1, z_2] \text{ has } d\vec{S} = \widehat{x}\, dy\, dz$$

$$S_3 : \quad y = y_1, (x, z) \in [x_1, x_2] \times [z_1, z_2] \text{ has } d\vec{S} = -\,\widehat{y}\, dx\, dz$$

$$S_4 : \quad y = y_2, (z, x) \in [z_1, z_2] \times [x_1, x_2] \text{ has } d\vec{S} = \widehat{y}\, dz\, dx$$

$$S_5 : \quad z = z_1, (y, x) \in [y_1, y_2] \times [x_1, x_2] \text{ has } d\vec{S} = -\,\widehat{z}\, dy\, dx$$

$$S_6 : \quad z = z_2, (x, y) \in [x_1, x_2] \times [y_1, y_2] \text{ has } d\vec{S} = \widehat{z}\, dx\, dy$$

The nice thing about the rectangular solid is that only one component of $\vec{F} = \langle P, Q, R \rangle$ cuts through a given face of the solid.

Observe that:

$$\begin{aligned}
\Phi_{12} &= \int_{S_1} \vec{F} \cdot d\vec{S} + \int_{S_2} \vec{F} \cdot d\vec{S} \qquad \text{(this defines } \Phi_{12} \text{ for future reference)} \\
&= \int_{y_1}^{y_2} \int_{z_1}^{z_2} \vec{F}(x_1, y, z) \cdot (-\,\widehat{x} dz\, dy) + \int_{z_1}^{z_2} \int_{y_1}^{y_2} \vec{F}(x_2, y, z) \cdot (\,\widehat{x} dz\, dy) \\
&= \int_{y_1}^{y_2} \int_{z_1}^{z_2} \big[-P(x_1, y, z)\big] dz\, dy + \int_{z_1}^{z_2} \int_{y_1}^{y_2} \big[P(x_2, y, z)\big] dy\, dz \\
&= \int_{z_1}^{z_2} \int_{y_1}^{y_2} \big[P(x_2, y, z) - P(x_1, y, z)\big] dy\, dz \\
&= \int_{z_1}^{z_2} \int_{y_1}^{y_2} \int_{x_1}^{x_2} \frac{\partial P}{\partial x}\, dx\, dy\, dz \qquad \text{by the FTC.}
\end{aligned}$$

Likewise,

$$\begin{aligned}
\Phi_{34} &= \int_{S_3} \vec{F} \cdot d\vec{S} + \int_{S_4} \vec{F} \cdot d\vec{S} \qquad \text{(this defines } \Phi_{34} \text{ for future reference)} \\
&= \int_{z_1}^{z_2} \int_{x_1}^{x_2} \vec{F}(x, y_1, z) \cdot (-\widehat{y} dx\, dz) + \int_{x_1}^{x_2} \int_{z_1}^{z_2} \vec{F}(x, y_2, z) \cdot (\widehat{y} dz\, dx) \\
&= \int_{z_1}^{z_2} \int_{x_1}^{x_2} \big[-Q(x, y_1, z)\big] dz\, dy + \int_{x_1}^{x_2} \int_{z_1}^{z_2} \big[Q(x, y_2, z)\big] dy\, dz \\
&= \int_{x_1}^{x_2} \int_{z_1}^{z_2} \big[Q(x, y_2, z) - Q(x, y_1, z)\big] dy\, dz \\
&= \int_{x_1}^{x_2} \int_{z_1}^{z_2} \int_{y_1}^{y_2} \frac{\partial Q}{\partial y}\, dx\, dy\, dz.
\end{aligned}$$

Repeating the same argument once more we derive:

$$\begin{aligned}
\Phi_{56} &= \int_{S_5} \vec{F} \cdot d\vec{S} + \int_{S_6} \vec{F} \cdot d\vec{S} \qquad \text{(this defines } \Phi_{12} \text{ for future reference)} \\
&= \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} \frac{\partial R}{\partial z}\, dz\, dy\, dx.
\end{aligned}$$

The flux of $\vec{F}$ over the entire boundary ∂E is found by summing the flux through each face. Therefore, by linearity of the triple integral for the second line,

$$\begin{aligned}
\iint_{\partial E} \vec{F} \cdot d\vec{S} &= \Phi_{12} + \Phi_{34} + \Phi_{56} \\
&= \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dz\, dy\, dx.
\end{aligned}$$

Which proves the divergence theorem for an arbitrary rectangular solid. □

In contrast to the earlier argument in this section the third dimension of the cylinder was not take as infinitesimal. That said, it wouldn't take much to modify the earlier argument for a finite height. The result just proved extends to more general solids in the way discussed earlier in this section following the cylindrical proof.