LECTURE 39 : THE FIELD TENSOR

GRIFFITHS takes some time to build this material with motivating examples. I'll take a different, less motivated, path with the aim of simply presenting what Griffiths motivates in its finished form.

Def^n/ The <u>4-potential</u> $\bar{A} = (V/c, \vec{A})$ is built with both the usual scalar potential and the vector potential. It is a contravariant 4-vector; $(A^\mu) = (V/c, A_x, A_y, A_z)$

Furthermore, we defined its <u>field tensor</u> via

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}$$

Notice x_μ is a <u>covariant</u> 4-vector thus $\bar{x}_{\mu'} = \Lambda^{\mu}_{\mu'} x_\mu$ and it follows that $\partial^\mu = \frac{\partial}{\partial x_\mu}$ is contravariant with $\bar{\partial}^{\mu'} = \Lambda^{\mu'}_{\mu} \partial^\mu$ whereas $\partial_\mu = \frac{\partial}{\partial x^\mu}$ has $\bar{\partial}_{\mu'} = \Lambda^{\mu}_{\mu'} \partial_\mu$.

Observe that $F^{\mu\nu}$ forms a <u>tensor</u> provided we are given A^μ is a contravariant 4-vector,

$$\begin{aligned}
\bar{F}^{\mu'\nu'} &= \bar{\partial}^{\mu'} \bar{A}^{\nu'} - \bar{\partial}^{\nu'} \bar{A}^{\mu'} \\
&= \Lambda^{\mu'}_{\mu} \partial^\mu (\Lambda^{\nu'}_{\nu} A^\nu) - \Lambda^{\nu'}_{\sigma} \partial^\sigma (\Lambda^{\mu'}_{\beta} A^\beta) \\
&= \Lambda^{\mu'}_{\mu} \Lambda^{\nu'}_{\nu} \partial^\mu A^\nu - \Lambda^{\mu'}_{\beta} \Lambda^{\nu'}_{\sigma} \partial^\sigma A^\beta \\
&= \Lambda^{\mu'}_{\mu} \Lambda^{\nu'}_{\nu} (\partial^\mu A^\nu - \partial^\nu A^\mu) \\
&= \Lambda^{\mu'}_{\mu} \Lambda^{\nu'}_{\nu} F^{\mu\nu}
\end{aligned}$$

Th^m/ Given A^μ is contravariant 4-vector; $\bar{A}^{\mu'} = \Lambda^{\mu'}_{\nu} A^\nu$ the field tensor is a rank 2 contravariant tensor;

$$\bar{F}^{\mu'\nu'} = \Lambda^{\mu'}_{\mu} \Lambda^{\nu'}_{\nu} F^{\mu\nu}$$

We know $\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$ and $\vec{B} = \nabla \times \vec{A}$
so we are not surprised $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$
gives components which are from $\vec{E}$ & $\vec{B}$,
$(x^\mu) = (ct, x, y, z)$ and $(x_\mu) = (-ct, x, y, z)$

$$F^{01} = -\frac{1}{c}\frac{\partial}{\partial t}(A_x) - \frac{\partial}{\partial x}(V/c) = \frac{1}{c}\left(-\nabla V - \frac{\partial \vec{A}}{\partial t}\right)_x = E_x/c$$

$$F^{02} = -\frac{1}{c}\frac{\partial}{\partial t}(A_y) - \frac{\partial}{\partial y}(V/c) = \frac{1}{c}\left(-\nabla V - \frac{\partial \vec{A}}{\partial t}\right)_y = E_y/c$$

$$F^{03} = -\frac{1}{c}\frac{\partial}{\partial t}(A_z) - \frac{\partial}{\partial z}(V/c) = \frac{1}{c}\left(-\nabla V - \frac{\partial \vec{A}}{\partial t}\right)_z = E_z/c$$

$$F^{12} = \frac{\partial}{\partial x}(A_y) - \frac{\partial}{\partial y}(A_x) = (\nabla \times \vec{A})_z = B_z$$

$$F^{13} = \frac{\partial}{\partial x}(A_z) - \frac{\partial}{\partial z}(A_x) = -(\nabla \times \vec{A})_y = -B_y$$

$$F^{23} = \frac{\partial}{\partial y}(A_z) - \frac{\partial}{\partial z}(A_y) = (\nabla \times \vec{A})_x = B_x$$

Notice $F^{\mu\nu} = -F^{\nu\mu}$ by its construction
thus $F^{00} = -F^{00} \Rightarrow F^{00} = 0$ and similarly
$F^{11} = F^{22} = F^{33} = 0$. There are 12 nonzero components,

$$\boxed{[F^{\mu\nu}] = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix}}$$

We would like to understand how Maxwell's Eq$^{\underline{n}}$s are expressed via the field tensor. Let's study the tensorial eq$^{\underline{n}}$ $\partial_\mu F^{\mu\nu}$ (well, expression for now)

$$\partial_\mu F^{\mu\nu} = \frac{\partial F^{0\nu}}{\partial x^0} + \frac{\partial F^{1\nu}}{\partial x^1} + \frac{\partial F^{2\nu}}{\partial x^2} + \frac{\partial F^{3\nu}}{\partial x^3}$$

$$= \frac{1}{c}\frac{\partial F^{0\nu}}{\partial t} + \frac{\partial F^{1\nu}}{\partial x} + \frac{\partial F^{2\nu}}{\partial y} + \frac{\partial F^{3\nu}}{\partial z}$$

$\nu = 0$

$$\partial_\mu F^{\mu 0} = \frac{-1}{c}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) = \frac{-1}{c}\nabla\cdot\vec{E}$$

$\nu = 1$

$$\partial_\mu F^{\mu 1} = \frac{1}{c}\frac{\partial}{\partial t}\left[E_x/c\right] + \frac{\partial}{\partial y}(-B_z) + \frac{\partial}{\partial z}(B_y) = \frac{1}{c^2}\frac{\partial E_x}{\partial t} - (\nabla\times\vec{B})_x$$

$\nu = 2$

$$\partial_\mu F^{\mu 2} = \frac{1}{c^2}\frac{\partial E_y}{\partial t} + \frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z} = \left(\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} - \nabla\times\vec{B}\right)_y$$

$\nu = 3$

$$\partial_\mu F^{\mu 3} = \frac{1}{c^2}\frac{\partial E_z}{\partial t} - \frac{\partial B_y}{\partial x} + \frac{\partial B_x}{\partial y} = \left(\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} - \nabla\times\vec{B}\right)_z$$

The expressions above must connect to Gauss' Eq$^{\underline{n}}$: $\nabla\cdot\vec{E} = \rho/\epsilon_0$ and Ampere's Law with Maxwell's Corr. $\nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$

a.k.a. $\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} - \nabla\times\vec{B} = -\mu_0\vec{J}$

$$\partial_\mu F^{\mu 0} = \frac{-1}{c}\nabla\cdot\vec{E} = \frac{-1}{c\,\epsilon_0}\rho = \mu_0\left(\frac{-1}{c\mu_0\epsilon_0}\rho\right) = -\mu_0 c\rho$$

$$\partial_\mu F^{\mu i} = \left(\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} - \nabla\times\vec{B}\right)_i = -\mu_0 J_i$$

If we use $-F^{\mu\nu} = F^{\nu\mu}$ then we have shown that

$\boxed{\partial_\mu F^{\nu\mu} = \mu_0 J^\nu}$ for $\boxed{J^\nu = (c\rho, \vec{J})}$

4-current.

We found Gauss Law and Ampere's Law with MAXWELL Correction can be elegantly summarized as the tensorial equation

$$\boxed{\partial_\nu F^{\mu\nu} = \mu_0 J^\mu}$$

where we've introduced the 4-current $J^\mu = (c\rho, \vec{J})$

It remains to express $\nabla \cdot \vec{B} = 0$ and $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ in a tensorial fashion. I know nice ways via differential forms ($F = dA$ so $dF = d(dA) = 0$ gives both) However, at the moment, I'm trying to stay in the general vicinity of Griffith's text. I need to explain his "dual" tensor $G^{\mu\nu}$

$$\partial_\mu G^{\nu\mu} = \frac{1}{c}\frac{\partial G^{\nu 0}}{\partial t} + \frac{\partial G^{\nu 1}}{\partial x} + \frac{\partial G^{\nu 2}}{\partial y} + \frac{\partial G^{\nu 3}}{\partial z}$$

$\underline{\nu = 0}$ $\quad \partial_\mu G^{0\mu} = \frac{\partial G^{01}}{\partial x} + \frac{\partial G^{02}}{\partial y} + \frac{\partial G^{03}}{\partial z} = \nabla \cdot \vec{B}$

$\underline{\nu = 1}$ $\quad \partial_\mu G^{1\mu} = \frac{1}{c}\frac{\partial G^{10}}{\partial t} + \frac{\partial G^{12}}{\partial y} + \frac{\partial G^{13}}{\partial z} = \left(-\frac{\partial \vec{B}}{\partial t} - \nabla \times \vec{E}\right)_1 \frac{1}{c}$

$\underline{\nu = 2}$ $\quad \partial_\mu G^{2\mu} = \frac{1}{c}\frac{\partial G^{20}}{\partial t} + \frac{\partial G^{21}}{\partial x} + \frac{\partial G^{23}}{\partial z} = \left(\frac{-\partial \vec{B}}{\partial t} - \nabla \times \vec{E}\right)_2 \frac{1}{c}$

$\underline{\nu = 3}$ $\quad \partial_\mu G^{3\mu} = \frac{1}{c}\frac{\partial G^{30}}{\partial t} + \frac{\partial G^{31}}{\partial x} + \frac{\partial G^{32}}{\partial y} = \left(\frac{-\partial \vec{B}}{\partial t} - \nabla \times \vec{E}\right)_3 \frac{1}{c}$

There is not a unique choice for $G^{\mu\nu}$ as above, but <u>if</u> we set $G^{01} = B_x$, $G^{02} = B_y$, $G^{03} = B_z$ then $G^{10} = -B_x$, $G^{20} = -B_y$, $G^{30} = -B_z$ need a $\frac{1}{c}$ ↗

then from $\nu = 1,2,3$ we deduce $G^{12} = -E_z/c$ and $G^{13} = E_y/c$ and $G^{23} = -E_x/c$

$$\boxed{\text{Def}^{\underline{n}}/ \; G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -\frac{E_z}{c} & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}}$$

We have shown that $\nabla \cdot \vec{B} = 0$ and Faraday's Law $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ may be expressed as $\boxed{\partial_\mu G^{\nu\mu} = 0}$ provided we define

$$G^{\mu\nu} = \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{bmatrix}$$

There is technically no need to introduce $G^{\mu\nu}$ since we could just as well express these <u>homogeneous</u> Maxwell Eq's as follows:

$$\boxed{\frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x^\mu} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu} = 0}$$

Notice there is no summation in the above, rather 3 free 4-indices $\Rightarrow$ 64 total equations. Many of these are not interesting. Problem 12.57 of Griffiths asked to show the eq$^{\underline{n}}$ above for $F_{\mu\nu}$ reproduces $\nabla \cdot \vec{B} = 0$ and $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$.

Notice,

$$\boxed{\text{Def}^{\underline{n}}/ \; F_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} F^{\alpha\beta}}$$

you'll need to find $F_{\mu\nu}$ explicitly to work through Problem 12.57.

DIFFERENTIAL FORMS APPROACH

We can write $F = dA$ where $A = A_\mu dx^\mu$

and $A_\mu = (-V/c, A_x, A_y, A_z)$ then

$$A = -V dt + \underbrace{A_x dx + A_y dy + A_z dz}_{\omega_{\vec{A}}}$$

Exterior calculus,

$$\begin{aligned}
dA &= -dV \wedge dt + dA_x \wedge dx + dA_y \wedge dy + dA_z \wedge dz \\
&= -\omega_{\nabla V} \wedge dt - \omega_{\frac{\partial \vec{A}}{\partial t}} \wedge dt + \Phi_{\nabla \times \vec{A}} \\
&= \omega_{-\nabla V - \frac{\partial \vec{A}}{\partial t}} \wedge dt + \Phi_{\nabla \times \vec{A}} \\
&= \omega_{\vec{E}} \wedge dt + \Phi_{\vec{B}} \\
&= \omega_{\vec{E}/c} \wedge (c\,dt) + \Phi_{\vec{B}} = (c\,dt) \wedge \omega_{-\vec{E}/c} + \Phi_{\vec{B}} \\
&= \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu
\end{aligned}$$

(see over, I find $F_{\mu\nu}$ explicitly)

Then,

$$\begin{aligned}
0 = d(dA) &= \frac{1}{2} dF_{\mu\nu} \wedge dx^\mu \wedge dx^\nu \\
&= \frac{1}{2} (\partial_\lambda F_{\mu\nu}) dx^\lambda \wedge dx^\mu \wedge dx^\nu \\
&= \frac{1}{6} (\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu}) dx^\lambda \wedge dx^\mu \wedge dx^\nu
\end{aligned}$$

The inhomogeneous Maxwell Eq$^{\underline{s}}$ are a bit trickier, we need Hodge Duality to formulate those. See Math 332.

$$F_{\mu\nu} = g_{\mu\alpha} F^{\alpha\beta} g_{\beta\nu}$$

$$(F_{\mu\nu}) = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 0 & F^{01} & F^{02} & F^{03} \\ F^{10} & 0 & F^{12} & F^{13} \\ F^{20} & F^{21} & 0 & F^{23} \\ F^{30} & F^{31} & F^{32} & 0 \end{bmatrix} \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 0 & F^{01} & F^{02} & F^{03} \\ -F^{10} & 0 & F^{12} & F^{13} \\ -F^{20} & F^{21} & 0 & F^{23} \\ -F^{30} & F^{31} & F^{32} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -F^{01} & -F^{02} & -F^{03} \\ -F^{10} & 0 & F^{12} & F^{13} \\ -F^{20} & F^{21} & 0 & F^{23} \\ -F^{30} & F^{31} & F^{32} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{bmatrix}$$

TRANSFORMING THE ELECTROMAGNETIC FIELD

If you believe the hype, $F^{\mu\nu}$ is a tensor thus it reveals how the electric and magnetic fields transform from (ct, x, y, z) frame and the $(c\bar{t}, \bar{x}, \bar{y}, \bar{z})$ frame

$$\boxed{\bar{F}^{\mu'\nu'} = \Lambda^{\mu'}_{\mu}\Lambda^{\nu'}_{\nu}F^{\mu\nu} = \Lambda^{\mu'}_{\mu}F^{\mu\nu}\Lambda^{\nu'}_{\nu}}$$

I'd like to implement the formula above as a matrix equation since we're used to such multiplication. The issue is how to identify rows and columns

$$[\bar{F}^{\mu'\nu'}] = \begin{bmatrix} \bar{F}^{0'0'} & \bar{F}^{0'1'} & \bar{F}^{0'2'} & \bar{F}^{0'3'} \\ \bar{F}^{1'0'} & \bar{F}^{1'1'} & \bar{F}^{1'2'} & \bar{F}^{1'3'} \\ \bar{F}^{2'0'} & \bar{F}^{2'1'} & \bar{F}^{2'2'} & \bar{F}^{2'3'} \\ \bar{F}^{3'0'} & \bar{F}^{3'1'} & \bar{F}^{3'2'} & \bar{F}^{3'3'} \end{bmatrix}$$

row index $\bar{F}^{\mu'\nu'}$ column index

$$[\Lambda^{\mu'}_{\mu}] = \begin{bmatrix} \Lambda^{0'}_{0} & \Lambda^{0'}_{1} & \Lambda^{0'}_{2} & \Lambda^{0'}_{3} \\ \Lambda^{1'}_{0} & \Lambda^{1'}_{1} & \Lambda^{1'}_{2} & \Lambda^{1'}_{3} \\ \Lambda^{2'}_{0} & \Lambda^{2'}_{1} & \Lambda^{2'}_{2} & \Lambda^{2'}_{3} \\ \Lambda^{3'}_{0} & \Lambda^{3'}_{1} & \Lambda^{3'}_{2} & \Lambda^{3'}_{3} \end{bmatrix}$$

$\Lambda^{\mu'}$ ← row index

Λ_{μ} ← column index

I should double check against our previous presentation of $\Lambda \in SO(1,3)$ as it related to $\bar{x}^{\mu'} = \Lambda^{\mu'}_{\nu}x^{\nu}$

μ'-th component of column vector $\bar{x}$

$$(\Lambda x)^{\mu'} = \underbrace{\sum_{\nu}\Lambda^{\mu'}_{\nu}x^{\nu}}_{\text{dot-product of } \mu'\text{-th row with } x.}$$

In standard linear algebra where A, B, C are matrices with components A_{ij}, B_{jk}, C_{kl} we'd define $(AB)_{ik} = \sum_j A_{ij} B_{jk}$ and $(BC)_{jl} = \sum_k B_{jk} C_{kl}$

and $(ABC)_{il} = \sum_j A_{ij} (BC)_{jl} = \sum_j A_{ij} \left(\sum_k B_{jk} C_{kl} \right)$

therefore, $(ABC)_{il} = \sum_j \sum_k A_{ij} B_{jk} C_{kl}$

row column row column row column

$$\bar{F}^{\mu'\nu'} = \Lambda^{\mu'}{}_{\mu} F^{\mu\nu} \Lambda^{\nu'}{}_{\nu}$$

need to transpose.

$$(\Lambda^T)^{\nu}{}_{\nu'} = \Lambda^{\nu'}{}_{\nu}$$

$$\boxed{\bar{F} = \Lambda F \Lambda^T}$$

<u>Remark</u>: the boosts in coordinate directions have $\Lambda = \Lambda^T$ so for our main application this nuance doesn't matter.

Let's check the matrix formula against the x-boost,

$$\bar{F} = \Lambda F \Lambda^T = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix} \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{c|c|c|c} -\beta\gamma E_x & \gamma E_x & E_y & E_z \\ -\gamma E_x & \beta\gamma E_x & B_z & -B_y \\ -\gamma E_y + \beta\gamma B_z & \beta\gamma E_y - \gamma B_z & 0 & B_x \\ -\gamma E_z - \beta\gamma B_y & \beta\gamma E_z + \gamma B_y & -B_x & 0 \end{array}\right]$$

$$= \left[\begin{array}{c|c|c|c} (-\beta\gamma^2 + \beta\gamma^2) E_x & \gamma^2(1-\beta^2) E_x & \gamma E_y - \beta\gamma B_z & \gamma E_z + \beta\gamma B_y \\ \gamma^2(\beta^2 - 1) E_x & (-\beta\gamma^2 + \beta\gamma^2) E_x & -\beta\gamma E_y + \gamma B_z & \beta\gamma E_z - \gamma B_y \\ \gamma(-E_y + \beta B_z) & \gamma(\beta E_y - B_z) & 0 & B_x \\ \gamma(-E_z - \beta B_y) & \gamma(B_y + \beta E_z) & -B_x & 0 \end{array}\right]$$

$$= \begin{bmatrix} 0 & E_x & \gamma(E_y - \beta B_z) & \gamma(E_z + \beta B_y) \\ -E_x & 0 & \gamma(B_z - \beta E_y) & -\gamma(B_y - \beta E_z) \\ -\gamma(E_y - \beta B_z) & -\gamma(B_z - \beta E_y) & 0 & B_x \\ -\gamma(E_z + \beta B_y) & \gamma(B_y + \beta E_z) & -B_x & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \bar{E}_{x'} & \bar{E}_{y'} & \bar{E}_{z'} \\ -\bar{E}_{x'} & 0 & \bar{B}_{z'} & -\bar{B}_{y'} \\ -\bar{E}_{y'} & -\bar{B}_{z'} & 0 & \bar{B}_{x'} \\ -\bar{E}_{z'} & \bar{B}_{y'} & -\bar{B}_{x'} & 0 \end{bmatrix}$$

I've omitted c in the E/c terms for brevity, I'll put them back

I'll focus on the components (0,1), (0,2), (0,3) (2,3), (3,1) and (1,2) to deduce the transformation laws,

$$\bar{E}_{x'}/c = E_x/c$$

$$\bar{E}_{y'}/c = \gamma(E_y/c - \beta B_z)$$

$$\bar{E}_{z'}/c = \gamma(E_z/c + \beta B_y)$$

$$\bar{B}_{x'} = B_x$$

$$\bar{B}_{y'} = \gamma(B_y + \beta E_z/c)$$

$$\bar{B}_{z'} = \gamma(B_z - \beta E_y/c)$$

Here $\beta = v/c$ therefore, $\beta c = v$,

$$\boxed{\begin{aligned} \bar{E}_{x'} &= E_x \\ \bar{E}_{y'} &= \gamma(E_y - vB_z) \\ \bar{E}_{z'} &= \gamma(E_z + vB_y) \\ \bar{B}_{x'} &= B_x \\ \bar{B}_{y'} &= \gamma(B_y + vE_z/c^2) \\ \bar{B}_{z'} &= \gamma(B_z - vE_y/c^2) \end{aligned}}$$

- These are the transformation laws which Griffiths derives from physical arguments of particular examples in Chapter 12. See Eq$^{\underline{n}}$ 12.111

If we define x to be $\parallel$-direction and y, z to be $\perp$ then $\bar{E}_{\parallel} = E_{\parallel}$ & $\bar{B}_{\parallel} = B_{\parallel}$

$$\bar{E}_{\perp} = \gamma(E_{\perp} + \vec{v} \times B_{\perp}) \quad \text{and} \quad \bar{B}_{\perp} = \gamma\left(B_{\perp} - \frac{1}{c^2}\vec{v} \times E_{\perp}\right)$$