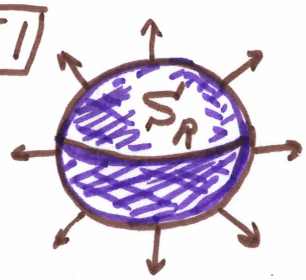
LECTURE 3: EXAMPLES OF VECTOR CALCULUS

Here I hope to illustrate *how* to calculate with the cylindrical and spherical frames and we should also study the DIRAC DELTA and Helmholtz Th$^{\underline{m}}$ which sets the stage for uniqueness of the voltage.

E1



$$d\vec{a} = (R^2 \sin\theta \, d\theta \, d\phi)\hat{r}$$

$$\vec{F} = r^n \hat{r}$$

$$\int_{S_R} \vec{F} \cdot d\vec{a} = \int_0^{2\pi}\int_0^{\pi} (R^n \hat{r}) \cdot (R^2 \sin\theta \, d\theta \, d\phi)\hat{r}$$

$$= R^{n+2}\left(\int_0^{2\pi} d\phi\right)\left(\int_0^{\pi} \sin\theta \, d\theta\right)$$

$$= 4\pi R^{n+2}$$

Remark: when $n = -2$ the $\Phi_{\vec{F}} = 4\pi$ for *all* R.

$$\nabla \cdot \vec{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 F_r\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta \, F_\theta\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left[F_\phi\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^{n+2}\right] \quad \text{since } F_r = r^n,\ F_\theta = F_\phi = 0.$$

$$= \frac{1}{r^2}(n+2)r^{n+1}$$

$$= (n+2)r^{n-1}$$

If $n-1 < 0$ then $\nabla \cdot \vec{F}$ blows-up as $r \to 0$. If $n < 1$ then $\nabla \cdot \vec{F} \to \infty$ as $r \to 0$.

(**Remark**: $\nabla \cdot \vec{F} = 0$ for $r \neq 0$ when $n = -2$.)

$\boxed{E1}$ continued, B_R ball with $\partial B_R = S_R$

$$\int_{B_R} (\nabla \cdot \vec{F})\, d\tau = \int_{B_R} (n+2) r^{n-1} r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} (n+2)\, r^{n+1} \sin\theta \, dr\, d\theta\, d\phi$$

$$= (n+2)(2\pi)(2)\left(\frac{R^{n+2}}{n+2}\right) \quad \text{for } n+2>0$$

$$= \frac{4\pi (n+2) R^{n+2}}{(n+2)}$$

$$= \underline{4\pi R^{n+2}}.$$

$\Big]$ assuming $n \neq -2$.

Remark: $\displaystyle\int_{S_R = \partial B_R} \vec{F} \cdot d\vec{a} = \int_{B_R} (\nabla \cdot \vec{F})\, d\tau = 4\pi R^{n+2}$

So, <u>ignoring</u> fact $\nabla \cdot \vec{F} = (n+2) r^{n-1}$ blows up for $n < 1$, it seems only $n = -2$ actually "fails" in the sense $\displaystyle\int_{B_R} (\nabla \cdot \vec{F})\, d\tau = 0$

If $n+2<0$ then,

$$\int_{B_R} (\nabla \cdot \vec{F})\, d\tau = 4\pi (n+2) \int_0^R r^{n+1}\, dr$$

$$= 4\pi (n+2) \lim_{\varepsilon \to 0^+} \int_\varepsilon^R r^{n+1}\, dr$$

$$= 4\pi (n+2) \lim_{\varepsilon \to 0^+} \left[\frac{R^{n+2}}{n+2} - \frac{\varepsilon^{n+2}}{n+2}\right]$$

$$= 4\pi \lim_{\varepsilon \to 0^+} \left(R^{n+2} - \varepsilon^{n+2}\right) = -\infty \neq \overbrace{\int_{S_R} \vec{F} \cdot d\vec{a}}^{4\pi R^{n+2}}$$

DIRAC DELTA DEVICE

We need to introduce the DIRAC DELTA because the concept of a finite charge at an infinitesmally small point forces an infinite divergence. We need the DIRAC DELTA to manage ∞. Begin with one-dimensional case,

Def$^{\underline{n}}$/ $\delta(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ \infty & \text{if } x = 0. \end{cases}$

For any function on $\mathbb{R}$, $\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)$

Likewise, $\int_{-\infty}^{\infty} f(x)\,\delta(x-x_0)\,dx = f(x_0)$

Equality of "delta functions" is characterized by integration

Def$^{\underline{n}}$/ D_1 and D_2 are equal delta functions provided $\int_{-\infty}^{\infty} f(x)\,D_1(x)\,dx = \int_{-\infty}^{\infty} f(x)\,D_2(x)\,dx$ for all "ordinary" functions $f(x)$

E2 Suppose $k \neq 0$ is a constant then we can demonstrate that $\delta(kx) = \frac{1}{|k|}\delta(x)$

$\underline{k>0}$
$$\int_{-\infty}^{\infty} f(x)\,\delta(kx)\,dx = \int_{-\infty}^{\infty} f\left(\frac{y}{k}\right)\delta(y)\frac{dy}{k} \qquad \left\{ \begin{array}{l} y = kx, \quad x = \frac{y}{k} \\ dy = k\,dx, \\ \text{bounds stay } \pm\infty \end{array} \right.$$

$$= \frac{f(0/k)}{k}$$

$$= \frac{f(0)}{k} = \int_{-\infty}^{\infty} f(x)\,\frac{\delta(x)}{k}\,dx.$$

(Griffiths discusses $k<0$ case, it's just substitution)

E3 $$\int_{-\infty}^{\infty} e^{3x}\,\delta(2x)\,dx = \int_{-\infty}^{\infty} e^{3x}\,\frac{\delta(x)}{2}\,dx = \frac{e^{3(0)}}{2} = \frac{1}{2}.$$

E4 $$\int_{-\infty}^{\infty} x^2\,\delta(3x-1)\,dx = \int_{-\infty}^{\infty} x^2\,\delta\left(3\left(x-\tfrac{1}{3}\right)\right)dx$$
$$= \int_{-\infty}^{\infty} x^2\,\frac{1}{3}\,\delta\left(x-\tfrac{1}{3}\right)dx$$
$$= \frac{1}{3}\left(\frac{1}{3}\right)^2$$
$$= \frac{1}{27}.$$

We're actually interested in higher-dimensional δ's

$$\boxed{\begin{array}{l} \text{Def}^{\underline{n}}/ \quad \displaystyle\int_{\mathbb{R}^n} f(\vec{r})\,\delta^n(\vec{r}-\vec{a})\,d^n x = f(\vec{a}) \\ \qquad\qquad \displaystyle\int_{\mathbb{R}^3} f(\vec{r})\,\delta^3(\vec{r}-\vec{a})\,d\tau = f(\vec{a}) \end{array}}$$

E5 $$\rho(\vec{r}) = Q\,\delta(\vec{r}-\vec{r}_o)$$
$$\int_{\mathcal{V}} \rho(\vec{r})\,d\tau = \int_{\mathcal{V}} Q\,\delta(\vec{r}-\vec{r}_o)\,d\tau = \begin{cases} Q & \text{if } \vec{r}_o \in \mathcal{V} \\ 0 & \text{if } \vec{r}_o \notin \mathcal{V} \end{cases}$$

$$\boxed{\text{Def}^{\underline{n}}/ \; \rho = \frac{dQ}{dV} \text{ is the electric charge density}}$$

Remark: the DIRAC-DELTA gives density of point charge.

Dirac Delta in Three Dimensions:

We saw the Coulomb field $\vec{F} = \frac{\hat{r}}{r^2}$ has the interesting feature that $\nabla \cdot \vec{F} = \begin{cases} \infty & \text{if } \vec{r} = 0 \\ 0 & \text{if } \vec{r} \neq 0 \end{cases}$

and $\int_{S_R} \vec{F} \cdot d\vec{a} = 4\pi$ for the sphere of radius R centered at the origin.

Griffiths 5^{th} Ed., pg. 48

" we found the divergence of $\frac{\hat{r}}{r^2}$ is zero everywhere except origin, and yet its integral over any volume containing the origin is constant (4π) "

Remark: the quote above has a problem, it expresses what we want to happen as opposed to what is. When $\nabla \cdot \left(\frac{\hat{r}}{r^2}\right) = \infty$ at $\vec{r} = 0$ this means we cannot even calculate $\int_{\mathcal{V}} \nabla \cdot \left(\frac{\hat{r}}{r^2}\right) d\tau$ if $\mathcal{V}$ contains $\vec{r} = 0$. Yet, Griffiths claims it is known to be 4π.

If we extend our concept of integration to include distributions (like the Dirac-Delta) then we want

$$\int_{\mathcal{V}} \nabla \cdot \left(\frac{\hat{r}}{r^2}\right) d\tau = \begin{cases} 0 & \text{if } 0 \notin \mathcal{V} \\ 4\pi & \text{if } 0 \in \mathcal{V} \end{cases}$$

Continuing,

$$\int_{\mathcal{V}} \nabla\cdot\left(\frac{\hat{r}}{r^2}\right) d\tau = \begin{cases} 0 & \text{if } 0 \notin \mathcal{V} \\ 4\pi & \text{if } 0 \in \mathcal{V} \end{cases} = \int_{\mathcal{V}} 4\pi\,\delta^3(\vec{r})\,d\tau$$

$$\therefore \quad \boxed{\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\,\delta^3(\vec{r})}$$

Example 1.16 of Griffith's $\quad B_R = \{(x,y,z) \mid x^2+y^2+z^2 \leq R^2\}$

$$J = \int_{\mathcal{V}} (r^2+2)\,\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) d\tau = \int_{\mathcal{V}=B_R} (r^2+2)\,4\pi\,\delta^3(\vec{r})\,d\tau = (0^2+2)4\pi = \boxed{8\pi}$$

Let's see another way to illustrate I.B.P. for surface integrals,

$$\nabla\cdot\left((r^2+2)\frac{\hat{r}}{r^2}\right) = (r^2+2)\,\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) + \nabla(r^2+2)\cdot\frac{\hat{r}}{r^2}$$

$$= (r^2+2)\,\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) + 2r\,\nabla r\cdot\frac{\hat{r}}{r^2} \quad , \quad \nabla r = \hat{r}$$

$$\therefore \quad (r^2+2)\,\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) = \nabla\cdot\left[(r^2+2)\frac{\hat{r}}{r^2}\right] - \frac{2}{r} \quad *$$

$$J = \int_{\mathcal{V}} (r^2+2)\,\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) d\tau = \int_{\mathcal{V}} \left(\nabla\cdot\left[(r^2+2)\frac{\hat{r}}{r^2}\right] - \frac{2}{r}\right) d\tau$$

$$= \int_{\partial\mathcal{V}} (r^2+2)\frac{\hat{r}}{r^2}\cdot(r^2\sin\theta\,d\theta\,d\phi)\hat{r} - \int_{\mathcal{V}} \frac{2\,d\tau}{r}$$

continued,

$$J = \int_{\partial V = S_R} (r^2+2)\frac{\hat{r}}{r^2} \cdot (r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}) - \int_{B_R} \frac{2}{r} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi} (R^2+2)\sin\theta \, d\phi - \int_0^R\int_0^{2\pi}\int_0^{\pi} 2r\sin\theta \, d\theta \, d\phi \, dr$$

$$= 4\pi(R^2+2) - \underbrace{r^2\Big|_0^R}_{R^2} \underbrace{\phi\Big|_0^{2\pi} -\cos\theta\Big|_0^{\pi}}_{4\pi}$$

$$= 4\pi R^2 + 8\pi - 4\pi R^2$$

$$= \boxed{8\pi.}$$

Remark: this calculation implicits an improper integration. Notice $\frac{1}{r}$ blows-up at $r=0$. Not to mention the divergent divergence...

THE HELMHOLTZ THEOREM

Section 7.9, pages 384 – 393 give statement and proof of this Thm and its application to E&M. Thm 7.9.7 from my Math 231 course supermath.info/CalculusIIIfall2025.pdf

Thm (HELMHOLTZ)

Suppose $\vec{F}$ is a vector field for which $\nabla \cdot \vec{F} = D$ and $\nabla \times \vec{F} = \vec{C}$. Futhermore, suppose $\vec{F} \longrightarrow 0$ as $\|\vec{r}\| \longrightarrow \infty$ and C & D tend to zero faster than $\frac{1}{\|\vec{r}\|^2}$ then $\vec{F}$ is uniquely given by

$$\vec{F} = -\nabla U + \nabla \times \vec{W}$$

where

$$U(\vec{r}) = \frac{1}{4\pi} \iiint_{\mathbb{R}^3} \frac{D(\vec{r}\,')}{\|\vec{r} - \vec{r}\,'\|} \, dV'$$

$$\vec{W}(\vec{r}) = \frac{1}{4\pi} \iiint_{\mathbb{R}^3} \frac{\vec{C}(\vec{r}\,')}{\|\vec{r} - \vec{r}\,'\|} \, dV'$$

- We'll ultimately see this Thm supports the legitimacy of the voltage or electric potential V and the vector potential $\vec{A}$ for which

$$\vec{E} = -\nabla \phi + \frac{\partial \vec{A}}{\partial t} \quad \text{and} \quad \vec{B} = \nabla \times \vec{A}$$

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7.9 green's identities and helmholtz theorem

This section is mostly developed in your homework, well, except for Green's Third identity and its applications. I think you'll be happy I did not relagate that to your homework. The purpose of this section is severalfold: (1.) we get to see how to use the divergence theorem yields further identities, in some sense these are the generalization of integration by parts to our current context, (2.) we lay some foundational mathematics which is important for the logical consistency of the potential formulation of electromagnetism (3.) we see new and fun calculations.

In your homework I asked you to show the following identities:

Proposition 7.9.1. *Green's First and Second Identities*

Suppose $f, g \in C^2(D')$ where E' is an open set containing the simple solid E which has piecewise smooth boundary ∂E. Then,

$$(1.)\quad \iiint_E \nabla f \cdot \nabla g \, dV + \iiint_E f\nabla^2 g \, dV = \iint_{\partial E} (f\nabla g) \cdot d\vec{S}$$

$$(2.)\quad \iiint_E (f\nabla^2 g - g\nabla^2 f)\, dV = \iint_{\partial E} (f\nabla g - g\nabla f) \cdot d\vec{S}$$

The identities above yield important results about harmonic functions. A function f is called **harmonic** on E if $\nabla^2 f = 0$ on E. You are also asked to show in the homework that:

Proposition 7.9.2. *properties of harmonic functions on a simple solid E*

Suppose the simple solid E which has piecewise smooth boundary ∂E and suppose f satisfies $\nabla^2 f = 0$ throughout E. Then,

(1.) $\iint_{\partial E} \nabla f \cdot d\vec{S} = 0$

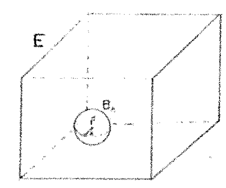
(2.) $\iiint_E \nabla f \cdot \nabla f \, dV = \iint_{\partial E} (f\nabla f) \cdot d\vec{S}$

(3.) $f(x, y, z) = 0$ for all $(x, y, z) \in \partial E \Rightarrow f(x, y, z) = 0$ for all $(x, y, z) \in E$.

(4.) If $\nabla^2 V_1 = b$ and $\nabla^2 V_2 = b$ throughout E and $V_1 = V_2$ on ∂E then $V_1 = V_2$ throughout E.

In words, (3.) states that if the restriction of f to ∂E is identically zero then f is zero throughout E. Whereas, (4.) states the solution to the **Poisson Equation** $\nabla^2 V = b$ is uniquely determined by its values on the boundary of a simple solid region.

This picture should help make sense of the Lemmas use to prove the Third Identity of Green:



Proposition 7.9.3. *Green's Third Identity.*

Suppose E is a simple solid, with piecewise smooth boundary ∂E, and assume f is twice differentiable throughout E. We denote $\vec{r} = (x, y, z)$ as a fixed, but arbitrary, point in E and denote the variables of the integration by $\vec{r}' = (x', y', z')$, so $dV' = dx'dy'dz'$ and $\nabla' = \hat{x}'\frac{\partial}{\partial x'} + \hat{y}'\frac{\partial}{\partial y'} + \hat{z}'\frac{\partial}{\partial z'}$. With this notation in mind,

$$f(\vec{r}) = \frac{-1}{4\pi}\iiint_E \frac{\nabla'^2 f(\vec{r}')}{||\vec{r} - \vec{r}'||} dV' + \frac{1}{4\pi}\iint_{\partial E}\left(-f(\vec{r}')\nabla'\left[\frac{1}{||\vec{r} - \vec{r}'||}\right] + \frac{\nabla' f(\vec{r}')}{||\vec{r} - \vec{r}'||}\right) \cdot d\vec{S}$$

We partly follow the proof in Colley. Begin with a Lemma,

Lemma 7.9.4. *Let h be a continuous function and S_R is a sphere of radius R centered at $\vec{r}$ then*

$$\lim_{R \to 0^+} \iint_{S_R} \frac{h(\vec{r}')}{||\vec{r} - \vec{r}'||} dS = 0.$$

The proof relies on the fact that if $\vec{r}' \in S_R$ then, by the definition of a sphere of radius R centered at $\vec{r}$, we have $||\vec{r} - \vec{r}'|| = R$. Thus,

$$\iint_{S_R} \frac{h(\vec{r}')}{||\vec{r} - \vec{r}'||} dS = \iint_{S_R} \frac{h(\vec{r}')}{||\vec{r} - \vec{r}'||} dS = \frac{1}{R}\iint_{S_R} h(\vec{r}')\, dS$$

Since h is continuous[27] it follows that there exist $\vec{a}, \vec{b} \in S_R$ such that $h(\vec{a}) \leq h(\vec{r}') \leq h(\vec{b})$ for all $\vec{r}\,' \in S_R$. Consequently,

$$\frac{h(\vec{a})}{||\vec{r} - \vec{r}'||} \leq \frac{h(\vec{r}')}{||\vec{r} - \vec{r}'||} \leq \frac{h(\vec{b})}{||\vec{r} - \vec{r}'||} \quad \Rightarrow \quad 4\pi R h(\vec{a}) \leq \iint_{S_R} \frac{h(\vec{r}')}{||\vec{r} - \vec{r}'||} dS \leq 4\pi R h(\vec{b})$$

As $R \to 0^+$ is clear that $\vec{a}, \vec{b} \to \vec{r}$ hence $h(\vec{b}) \to h(\vec{r})$ and $h(\vec{a}) \to h(\vec{r})$. Observe, as $R \to 0^+$ we obtain $4\pi R h(\vec{a}) \to 4\pi R h(\vec{r}) \to 0$ and $4\pi R h(\vec{b}) \to 4\pi R h(\vec{r}) \to 0$. The Lemma above follows by the squeeze theorem ∇

[27] I use the extreme value theorem: any continuous, real-valued, image of a compact domain attains its extrema;

Lemma 7.9.5. *Let h be a continuous function and S_R is a sphere of radius R centered at $\vec{r}$ then*

$$\lim_{R \to 0^+} \iint_{S_R} h(\vec{r}')\nabla'\left[\frac{1}{||\vec{r} - \vec{r}'||}\right] \cdot d\vec{S} = -4\pi h(\vec{r}).$$

The proof of this Lemma is similar to the previous. We begin by simplifying the integral. Note,

$$\vec{r} - \vec{r}' = (x - x',\ y - y',\ z - z')$$

Let $L = ||\vec{r} - \vec{r}'||$ thus $L^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$. Calculate,

$$\frac{\partial}{\partial x'}[L^2] = 2L\frac{\partial L}{\partial x'} = -2(x - x') \quad \Rightarrow \quad \frac{\partial L}{\partial x'} = \frac{x' - x}{L}$$

Likewise, a similar calculation shows, $\frac{\partial L}{\partial y'} = \frac{y' - y}{L}$ and $\frac{\partial L}{\partial z'} = \frac{z' - z}{L}$. Thus,

$$\nabla'\left[\frac{1}{||\vec{r} - \vec{r}'||}\right] = \nabla'\left[\frac{1}{L}\right] = -\frac{1}{L^2}\nabla' L = \frac{1}{L^3}(x - x',\ y - y',\ z - z') = \frac{1}{L^3}(\vec{r} - \vec{r}')$$

Continuing, note the normal vector field $\vec{N}$ on S_R points in the $\vec{r} - \vec{r}'$ direction at $\vec{r}\,'$ thus

$$d\vec{S} = \frac{\vec{r} - \vec{r}'}{||\vec{r} - \vec{r}'||} dS = \frac{1}{L}(\vec{r} - \vec{r}')dS.$$

To calculate the surface integral, note $L = R$ on S_R thus,

$$\begin{aligned}\iint_{S_R} h(\vec{r}')\nabla'\left[\frac{1}{||\vec{r} - \vec{r}'||}\right] \cdot d\vec{S} &= \iint_{S_R} h(\vec{r}')\frac{1}{R^3}(\vec{r} - \vec{r}') \cdot \frac{1}{R}(\vec{r} - \vec{r}')dS \\ &= \iint_{S_R} h(\vec{r}')\frac{||\vec{r} - \vec{r}'||^2}{R^4} dS \\ &= \frac{1}{R^2}\iint_{S_R} h(\vec{r}')\, dS\end{aligned}$$

We once again use the squeeze theorem argument of the previous Lemma, since h is continuous it follows that there exist $\vec{a}, \vec{b} \in S_R$ such that $h(\vec{a}) \leq h(\vec{r}') \leq h(\vec{b})$ for all $\vec{r}\,' \in S_R$. Consequently,

$$\iint_{S_R} h(\vec{a})dS \leq \iint_{S_R} h(\vec{r}')dS \leq \iint_{S_R} h(\vec{b})dS$$

But, the integrals on the edges are easily calculated since $h(\vec{a}), h(\vec{b})$ are just constants and we deduce:

$$4\pi R^2 h(\vec{a}) \leq \iint_{S_R} h(\vec{r}')dS \leq 4\pi R^2 h(\vec{b}) \quad \Rightarrow \quad h(\vec{a}) \leq \frac{1}{4\pi R^2}\iint_{S_R} h(\vec{r}')dS \leq h(\vec{b}).$$

As $R \to 0^+$ is clear that $\vec{a}, \vec{b} \to \vec{r}$ hence $h(\vec{b}) \to h(\vec{r})$ and $h(\vec{a}) \to h(\vec{r})$ and the lemma follows by the squeeze theorem. ∇.

Green's Second Identity applies to solid regions with holes provided we give the boundary the standard outward orientation. With that in mind, consider $E' = E - B_R$ where B_R is the closed-ball of radius R which takes boundary S_R; $\partial B_R = S_R$. However, we insist that $\partial E' = \partial E \cup (-S_R)$ so the hole at $\vec{r}$ has inward-pointing normals. Apply Green's Second Identity with $g(\vec{r}') = \frac{1}{||\vec{r} - \vec{r}'||}$:

$$\iiint_{E'}\left(f(\vec{r}')\nabla'^2\left[\frac{1}{||\vec{r} - \vec{r}'||}\right] - \frac{\nabla'^2 f(\vec{r}')}{||\vec{r} - \vec{r}'||}\right) dV' = \iint_{\partial E'}\left(f(\vec{r}')\nabla'\left[\frac{1}{||\vec{r} - \vec{r}'||}\right] - \frac{\nabla' f(\vec{r}')}{||\vec{r} - \vec{r}'||}\right) \cdot d\vec{S}$$

Once more, use the notation $L = ||\vec{r} - \vec{r}'||$. Observe that

$$\frac{\partial^2 L}{\partial x'^2} = \frac{\partial}{\partial x'}\left[\frac{x' - x}{L}\right] = \frac{L - (x' - x)(\frac{x' - x}{L})}{L} = \frac{L^2 - (x' - x)^2}{L^2}$$

similar formulas hold for y and z hence:

$$\nabla'^2 \frac{1}{||\vec{r} - \vec{r}'||} = \frac{\partial^2 L}{\partial x'^2} + \frac{\partial^2 L}{\partial y'^2} + \frac{\partial^2 L}{\partial z'^2} = \frac{L^2 - (x' - x)^2 - (y' - y)^2 - (z' - z)^2}{L^2} = 0.$$

Therefore, Green's Second Identity simplifies slightly: (in the second line we use $\partial E' = \partial E \cup (-S_R)$)

$$\begin{aligned}\iiint_{E'}\left(-\frac{\nabla'^2 f(\vec{r}')}{||\vec{r} - \vec{r}'||}\right) dV' &= \iint_{\partial E'}\left(f(\vec{r}')\nabla'\left[\frac{1}{||\vec{r} - \vec{r}'||}\right] - \frac{\nabla' f(\vec{r}')}{||\vec{r} - \vec{r}'||}\right) \cdot d\vec{S} \\ &= \iint_{\partial E}\left(f(\vec{r}')\nabla'\left[\frac{1}{||\vec{r} - \vec{r}'||}\right] - \frac{\nabla' f(\vec{r}')}{||\vec{r} - \vec{r}'||}\right) \cdot d\vec{S} \\ &\quad - \iint_{S_R}\left(f(\vec{r}')\nabla'\left[\frac{1}{||\vec{r} - \vec{r}'||}\right] - \frac{\nabla' f(\vec{r}')}{||\vec{r} - \vec{r}'||}\right) \cdot d\vec{S}\end{aligned}$$

Observe that $\nabla' f(\vec{r}')$ is continuous as required by the Lemma 7.9.5. Suppose $R \to 0^+$ and apply Lemma 7.9.5 and Lemma 7.9.4 to simplify the surface integrals over S_R. Moreover, as $R \to 0^+$ we see $E' \to E - \{\vec{r}\}$ and it follows:

$$\iiint_E\left(-\frac{\nabla'^2 f(\vec{r}')}{||\vec{r} - \vec{r}'||}\right) dV' = \iint_{\partial E}\left(f(\vec{r}')\nabla'\left[\frac{1}{||\vec{r} - \vec{r}'||}\right] - \frac{\nabla' f(\vec{r}')}{||\vec{r} - \vec{r}'||}\right) \cdot d\vec{S} + 4\pi f(\vec{r}).$$

Green's Third Identity follows by algebra, just solve for $f(\vec{r})$. $\square$

I like the proof of this proposition because it is little more than careful calculation paired with a few natural limits. If you study the Coulomb field and the way it escapes the divergence theorem[28] due to the singularity at the origin then you might be led to these calculations. In any event, we've proved it so we can use it now.

Problem: Solve $\nabla^2 f = b$ for f on E.

Take the Laplacian of Green's Third Identity with respect to $\vec{r} \in E$. It can be shown through relatively straight-forward differentiation the surface integral over ∂E are trivial hence we find the beautiful formula:

$$\nabla^2 f(\vec{r}) = \frac{-1}{4\pi}\nabla^2 \iiint_E \frac{\nabla'^2 f(\vec{r}')}{||\vec{r} - \vec{r}'||} dV'$$

Let $h(\vec{r}) = \nabla^2 f(\vec{r})$ to see what the formula above really says:

$$\boxed{h(\vec{r}) = \nabla^2 \iiint_E \frac{-h(\vec{r}')}{4\pi||\vec{r} - \vec{r}'||} dV'}$$

[28] I discussed how many physics students are taught to escape the difficulty in Section 7.8.1

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We have only provided evidence it is true if h is the Laplacian of another function f, but it is true in more generality[29] The formula boxed above shows how a particular modified triple integration in an inverse process to taking the Laplacian. It's like a second-order FTC for volume integrals. Returning to the problem, and placing faith in the generality of the formula[30], think of $h = f$ and assume $\nabla^2 f = b$. (I leave the details of why ∇^2 can be pulled into the integral and changed to ∇'^2)

$$f(\vec{r}) = \nabla^2 \iiint_E \frac{-f(r'')}{4\pi||r' - r''||}dV' = \iiint_E \frac{-\nabla'^2 f(r'')}{4\pi||r' - r''||}dV' = \iiint_E \frac{-b(r'')}{4\pi||r' - r''||}dV'$$

Therefore, we find the following theorem.

Theorem 7.9.6. *Integral solution to Poisson's Equation:*

If $\nabla^2 f = b$ for some continuous function b on a simple solid region E then

$$\boxed{f(\vec{r}) = \iiint_E \frac{-b(\vec{r}')}{4\pi||\vec{r} - \vec{r}'||}dV'.}$$

If we are given that b tends to zero *fast* enough as we let $||\vec{r}|| \to \infty$ then the domain of integration E may be extended to $\mathbb{R}^3$ and the boxed equation serves to define a global solution to Poisson's Equation. Helmholtz' Theorem is related to this discussion. Let me state the Theorem for reference:

Theorem 7.9.7. *Helmholtz*

Suppose $\vec{F}$ is a vector field for which $\nabla \cdot \vec{F} = \vec{D}$ and $\nabla \times \vec{F} = \vec{C}$. Furthermore, suppose $\vec{F} \to 0$ as $||\vec{r}|| \to \infty$ and C, D tend to zero faster than $1/||\vec{r}||^2$ then $\vec{F}$ is uniquely given by:

$$\vec{F} = -\nabla U + \nabla \times \vec{W}$$

where

$$U(\vec{r}) = \frac{1}{4\pi}\iiint_{\mathbb{R}^3} \frac{D(\vec{r}')}{||\vec{r} - \vec{r}'||}dV' \quad \& \quad W(\vec{r}) = \frac{1}{4\pi}\iiint_{\mathbb{R}^3} \frac{C(\vec{r}')}{||\vec{r} - \vec{r}'||}dV'$$

For the interested reader, this does not contradict the multivariate Taylor theorem. In Taylor's theorem we are given all derivatives at a particular point and that data allows us to reconstruct the function. In Helmholtz Theorem we are given two globally defined vector fields and some first order derivative data which is sufficient to reconstruct the vector field $\vec{F}$. The difference is in the domain of the givens. At a point vs. on all points.

[29]Colley points to Kellog's *Foundations of Potential Theory* from 1928, I'd wager you could find dozens of texts to support this point. For example, Flander's text develops these ideas in blinding generality via differential form calculations.

[30]sorry folks, I'd like to fill this gap, but time's up

7.10 maxwell's equations and the theory of potentials

The central equations of electromagnetism are known as **Maxwell's Equations** in honor of James Clerk Maxwell who completed these equations around the time of the Civil War. Parts of these were known before Maxwell's work, however, Maxwell's addition of the term $\mu_o\epsilon_o\frac{\partial E}{\partial t}$ was crucial in the overal consistency and eventual success of the theory in explaining electromagnetic phenomena. In this section we will examine how these equations can either be stated locally as a system of PDEs or as integrals which yield the fields. Potentials for the fields are also analyzed, we see how Green's identities help connect the integral formulations of the potentials and the corresponding PDE which are, in a particular gauge, Poisson-type equations. Please understand I am not attempting to explain the physics here! That would take a course or two, our focus is on the mathematical backdrop for electromagnetism. I'll leave the physics for our junior-level electromagnetism course.

Let me set the stage here: $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field, both depend on time t and space x, y, z generally. In principle the particular field configuration is due to a given charge density ρ and current density $\vec{J}$. The electric and magnetic fields are solutions to the following set of PDEs[31]:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_o}, \qquad \nabla \times \vec{B} = \mu_o\left(\vec{J} + \epsilon_o\frac{\partial \vec{E}}{\partial t}\right), \qquad \nabla \cdot \vec{B} = 0, \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

These are Gauss' Law, Ampere's Law with Maxwell's correction, no magnetic monopoles and Faraday's Law all written as local field equations. In most elementary electromagnetics courses[32] these Laws are presented as integral relations. In those elementary treatments the integral relations are taken as primary, or basic, experimentally verified results. In constrast, to Colley's *Vector Calculus* text, or most introductory physics texts, I take the PDE form of Maxwell's equations as basic. In my opinion, these are the nexus from which all else flows in E & M. For me, Maxwell's equations *define* electromagnetism[33]. Let's see how Stokes' and Gauss' theorems allow us to translate Maxwell's equations in PDE-form to Maxwell's equations in integral form.

7.10.1 Gauss' Law

Suppose M is a simple solid with closed surface $S = \partial M$ where and apply the divergence theorem to Gauss' Law:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_o} \;\Rightarrow\; \iiint_M (\nabla \cdot \vec{E})dV = \frac{1}{\epsilon_o}\iiint_M \rho\, dV \;\Rightarrow\; \boxed{\iint_{\partial M} \vec{E} \cdot d\vec{S} = \frac{Q_{enc}}{\epsilon_o}}$$

The equation boxed above is what we call *Gauss' Law* in the freshman[34] E & M course. It simply states that the flux through a closed "gaussian" surface is given by the charge enclosed in the surface divided by the permitivity of space.

Example 7.10.1. *Suppose you have a very long line of charge along the z-axis with constant density $\lambda = dQ/dz$. Imagine a Gaussian cylinder S length L centered about the z-axis: only the*

[31]I use SI units and ϵ_o and μ_o are the permitivity and permeability of empty space

[32]for example, Physics 232 at LU

[33]ok, in truth there is a case that escapes Maxwell's equations, but that nonlinear case is considerably more sophisticated than these notes...

[34]because there are three more years of physics past this course... it is to be done in the Freshman year of university.