LECTURE 3: BUILD A BIJECTION

A given function or map $f : X \longrightarrow Y$ need not be a bijection. That said, to make a surjection we can replace Y with $\mathrm{range}(f) = f(X)$ then $f : X \longrightarrow f(X)$ is certainly onto. Now, how to adjust f to be an <u>injective</u> map?

Def^n/ $f : X \to Y$ is injective or 1-1 iff $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \quad \forall x_1, x_2 \in X$

Consider,

$$f^{-1}\{y\} = \{x \in X \mid f(x) = y\}$$

If $\exists\, x_1, x_2 \in X$ for which $f(x_1) = y = f(x_2)$ then $x_1, x_2 \in f^{-1}\{y\}$

Th^m/ A map is injective if for $f : X \to Y$ we have $f^{-1}\{y\} = \emptyset$ or $f^{-1}\{y\} = \{x\}$ for some $x \in X$ for each $y \in Y$. In other words, an injective map is one for which every non-empty fiber is a singleton

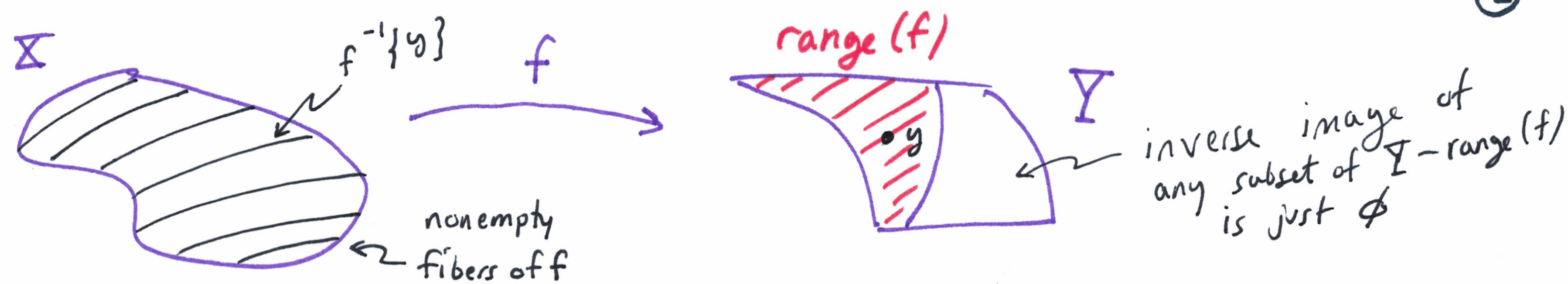
$\boxed{E1}$ $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x^2$

$f(x_1) = f(x_2)$

$x_1^2 = x_2^2$

$\Rightarrow x_1 = \pm x_2$ (not 1-1)

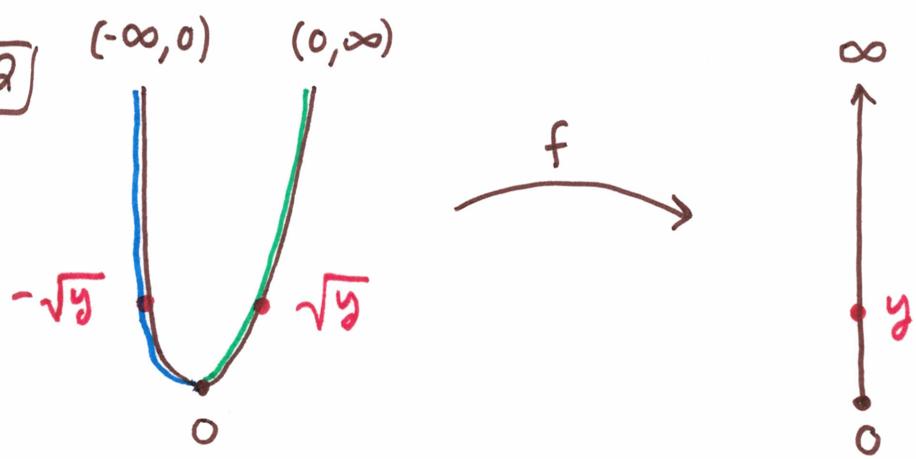
$$f^{-1}\{y\} = \begin{cases} \emptyset & \text{if } y < 0 \\ \{0\} & \text{if } y = 0 \\ \{-\sqrt{y}, \sqrt{y}\} & \text{if } y > 0 \end{cases}$$



$\left[\begin{array}{l} x_1 \sim x_2 \text{ iff } f(x_1) = f(x_2) \\ \text{defines an equivalence relation} \\ \text{on } X \text{ and the} \\ \text{fibers are the partition} \\ \text{of this relation} \end{array}\right]$

- To obtain an injection we need some method to <u>select</u> just one point from each nonempty fiber.

$\boxed{E2}$



(I bent the # line for convenience of picture)

$f : \mathbb{R} \to [0,\infty)$

$f(x) = x^2$ is <u>surjective</u>

$f^{-1}\{y\} = \{-\sqrt{y}, \sqrt{y}\}$

fibers are pairs for $y \neq 0$

Select $\sqrt{y}$ from each fiber to make <u>injection</u>

$f\big|_{[0,\infty)} : [0,\infty) \to [0,\infty)$

in fact, this is a bijection.

<u>Comment</u>: the adhoc method used in $\boxed{E2}$ to cut down the domain so the <u>restricted</u> function is injective is generally an art. which subset of the domain to use? That said, the <u>AXIOM OF CHOICE</u> indicates it is possible in principle.

$\boxed{E3}$ $f : \mathbb{R}^{3\times 3} \longrightarrow \mathbb{R}^3$ defined by $f(M) = col_3(M)$ is clearly onto.

Note $f^{-1}\left\{\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right\} = \left\{ \begin{bmatrix} M_{11} & M_{12} & a \\ M_{21} & M_{22} & b \\ M_{31} & M_{32} & c \end{bmatrix} \,\middle|\, M_{11}, M_{12}, M_{21}, M_{22}, M_{31}, M_{32} \in \mathbb{R} \right\}$

What restriction to place on $\mathbb{R}^{3\times 3}$ so $f^{-1}\left\{\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right\} \cap S$ is singleton?

$\exists \infty$ly choices, but setting $M_{1i}, M_{2i} = 0$ for $i = 1,2,3$ is nice.

$$S = \{ M \in \mathbb{R}^{3\times 3} \mid M_{ij} = 0 \text{ for } 1 \le i \le 2,\ 1 \le j \le 3 \}$$

$$f^{-1}\left\{\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right\} \cap S = \left\{ \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & c \end{bmatrix} \right\}$$

Then $f|_S : S \longrightarrow \mathbb{R}^3$ is a <u>bijection</u>.

<u>Geometrically</u> $\mathbb{R}^{3\times 3}$ is 9-dim'l space and the fibers are 6-dim'l slices. The set S is a 3-dim'l space which cuts through each fiber in just one pt. My picture on 2 is a mnemonic.

__Comment__: I'm covering Chapter 2 sections 2.2, 2.3, 2.4, 2.5, 2.6 rather lightly. It is good for you to read them in their entirety for your mathematical breadth. I'll record a few high points here:

- $\mathbb{R}$ is complete; every non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound.

- __Lemma 2.3__: Let $X \subseteq \mathbb{N}$ be an infinite subset. Then $\exists$ strictly inc. bijection $f : \mathbb{N} \to X$.

Def^n/ X, Y have same __cardinality__ if $\exists$ bijection $f : X \to Y$ and we write $|X| = |Y|$

Def^n/ X is countably infinite if $|X| = |\mathbb{N}|$ and X is countable if X is either finite or countably ∞.

- __Cantor's Th^m__: for $X \neq \emptyset$, $\nexists$ surjective maps $f : X \to \mathcal{P}(X)$. Moreover $|X| \neq |\mathcal{P}(X)|$

Def^n/ $\mathcal{P}(X) = \{A \mid A \subseteq X\} = \{\emptyset, \ldots, X\}$ the __power set__ of X

Def^n/ X^Y is set of all maps $f : X \to Y$

- $|\mathbb{R}| = |\mathbb{N}^{\mathbb{N}}| = |\mathbb{R}^{\mathbb{N}}| = |\mathcal{P}(\mathbb{N})|$

Axiom of Choice (V1)

If $g: Y \to X$ is onto map between sets, $\exists$ map $f: X \to Y$ s.t. $g(f(x)) = x \ \forall x \in X$

Defn/ An equivalence relation on X is a relation $\sim$ which is
- reflexive $x \sim x$
- symmetric $x \sim y \Rightarrow y \sim x$
- transitive $x \sim y$ and $y \sim z \Rightarrow x \sim z$

equivalence class $[x] = \{ y \in X \mid y \sim x \}$

quotient set $X/\sim \; = \{ [x] \mid x \in X \}$ set of $\sim$ classes.

quotient map $\pi : X \to X/\sim$ given by $\pi(x) = [x]$

• Lemma 2.18: Let $\sim$ be equivalence relation on X, $\pi : X \to X/\sim$ the quotient map and $f: X \to Y$ a mapping. TFAE,

1.) the map f is constant on equivalence classes; $f(x) = f(y)$ whenever $x \sim y$.

2.) $\exists!\ g: X/\sim \;\to Y$ such that $f = g \circ \pi$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & Y \\ X/\sim \arrow[ur, dotted, "g"'] & \end{tikzcd}$$

Axiom of Choice (V2)

Let $X = \bigcup \{ X_i \mid i \in I \}$ where $X_i \neq \emptyset \ \forall i$ in the index set I. Then $\exists$ a map $\underbrace{f: I \to X}_{\text{"choice function"}}$ such that $f(i) \in X_i$ for every i.

Thm/ (Zorn's Lemma) Let $(X, \leq)$ be nonempty ordered set. If every chain in X is upper bounded then X contains maximal elements.