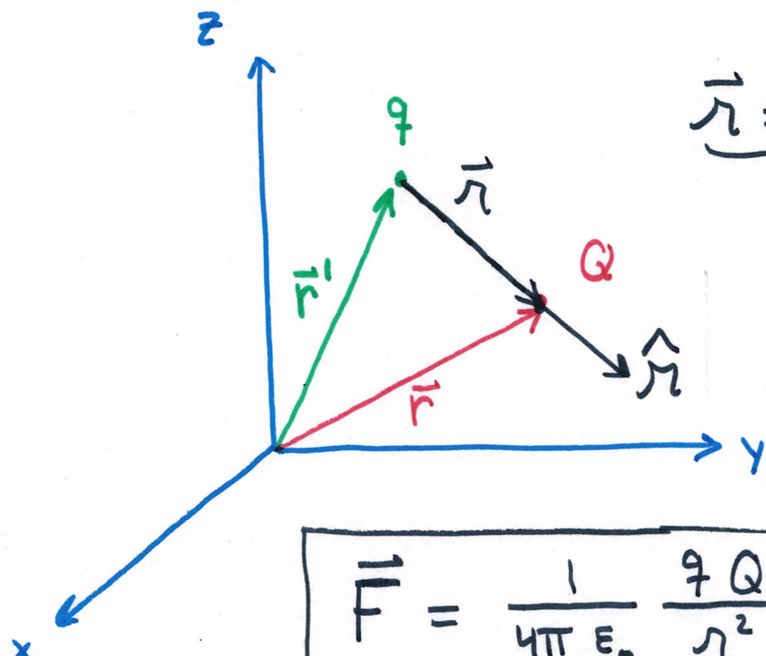
Lecture 4: Electric Field

Given some collection of source charges $q_1, q_2, q_3, \ldots$ what force do these exert on another charge Q. What trajectory will Q follow on the basis of the electric force due to the source charges?

- principle of superposition: net electric force on Q is sum of forces from $q_1, q_2, q_3, \ldots$

In Chapter 2 of Griffiths (and 3) we assume the source charges are fixed in place and that any variable electric field has gone away...
If the source charges move then we can write down force on Q, but that story is much more complicated (and beautiful, see p. 464) (eq: 10.72, eq: 10.74)



$$\underbrace{\vec{\mathscr{r}} = \vec{r} - \vec{r}'}_{\text{separation vector}}$$

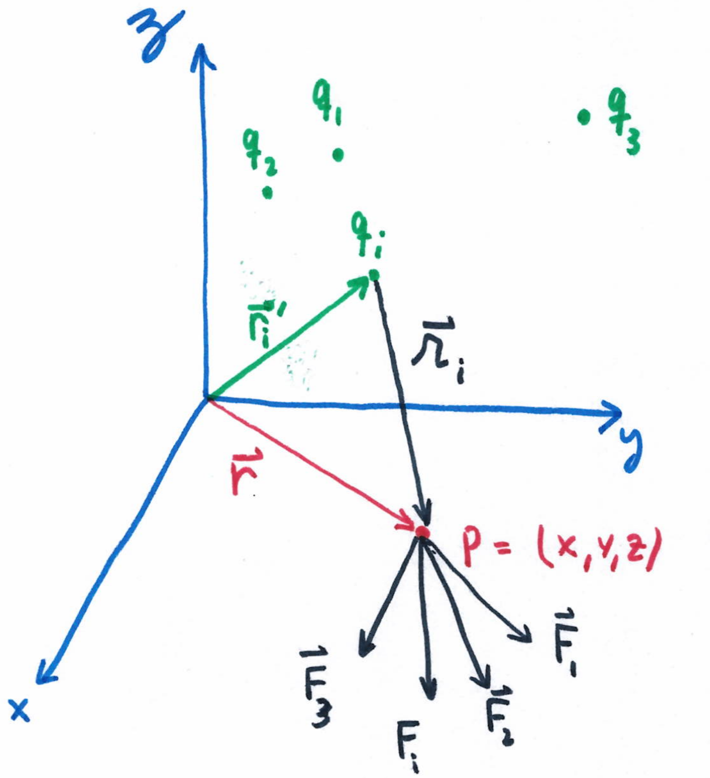
$\vec{r}$ = field point

$\vec{r}'$ = source point

$$\boxed{\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\mathscr{r}}}$$

$$\varepsilon_0 = 8.85 \times 10^{-12} \frac{C^2}{Nm^2}$$

Electric Field



For finite collection of point charges q_i at positions $\vec{r}_i$ for $1 \leq i \leq n$ we find total force $\vec{F}$ at P on Q as follows:

$$\vec{F} = \sum_{i=1}^{n} \vec{F}_i = \sum_{i=1}^{n} \frac{q_i Q}{4\pi\varepsilon_0} \frac{1}{\mathscr{r}_i^2} \hat{\mathscr{r}}_i$$

$$= Q \underbrace{\left(\sum_{i=1}^{n} \frac{q_i}{4\pi\varepsilon_0} \frac{\hat{\mathscr{r}}_i}{\mathscr{r}_i^2} \right)}_{\vec{E} \text{ at } \vec{r}}$$

Def^n/ Given a test charge Q at $\vec{r}$ if the net-electric force $\vec{F} = Q\vec{E}$ then we say $\vec{E}$ is the <u>electric field</u> at $\vec{r}$.

$\boxed{E1}$ for the finite collection above,

$$\vec{E}(\vec{r}) = \sum_{i=1}^{n} \frac{q_i}{4\pi\varepsilon_0} \frac{\hat{\mathscr{r}}_i}{\mathscr{r}_i^2}.$$

Continuous Charge Distributions

We argued for finite collection of source charges $q_1, .., q_n$ the electric field at $\vec{r}$ is given by

$$\vec{E}(\vec{r}) = \sum_{i=1}^{n} \frac{q_i}{4\pi\varepsilon_0} \frac{\hat{\mathscr{r}}_i}{\mathscr{r}_i^2}$$

If we allow $n \to \infty$ the above becomes a continuous sum

$$\boxed{\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{1}{\mathscr{r}^2} \hat{\mathscr{r}}\, dq}$$

integral form of Coulomb's Law for continuous distributions of charge

For $\underbrace{\lambda = \frac{dq}{d\ell}}_{\text{linear charge density}}$ or $\underbrace{\sigma = \frac{dq}{da}}_{\text{surface charge density}}$ or $\underbrace{\rho = \frac{dq}{d\tau}}_{\text{volume charge density}}$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_C \frac{1}{\mathscr{r}^2} \hat{\mathscr{r}}\, \lambda(\vec{r}\,')\, d\ell' \qquad \text{(line)(curve)}$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_S \frac{1}{\mathscr{r}^2} \hat{\mathscr{r}}\, \sigma(\vec{r}\,')\, da' \qquad \text{(surface)}$$

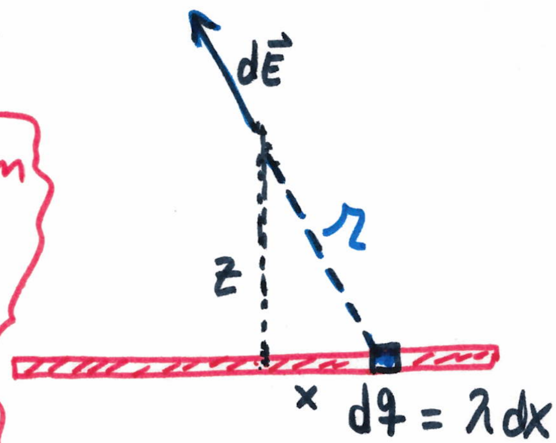
$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{1}{\mathscr{r}^2} \hat{\mathscr{r}}\, \rho(\vec{r}\,')\, d\tau' \qquad \text{(volume)}$$

Remark: calculation of such integrals can be tough.

4

E1

uniform density linear charge λ over $[-L, L]$



$d\vec{E} = \langle dE_x, dE_z \rangle$

I'll suppress y-notation for this example.

$$\left.\begin{aligned} \vec{r} &= \langle 0, z \rangle \\ \vec{r}\,' &= \langle x, 0 \rangle \end{aligned}\right\} \quad \vec{\mathcal{R}} = \vec{r} - \vec{r}\,' = \langle -x, z \rangle$$

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{dq}{\mathcal{R}^2} \hat{\mathcal{R}} = \frac{\lambda\, dx}{4\pi\varepsilon_0} \frac{1}{\mathcal{R}^3} \vec{\mathcal{R}} = \frac{\lambda \langle -x, z \rangle dx}{4\pi\varepsilon_0 (x^2+z^2)^{3/2}}$$

Now integrate over $-L \le x \le L$,

$$\vec{E}(0,z) = \frac{\lambda}{4\pi\varepsilon_0} \int_{-L}^{L} \frac{1}{(x^2+z^2)^{3/2}} \langle -x, z \rangle\, dx$$

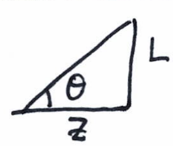
$$= \frac{\lambda}{4\pi\varepsilon_0} \left\langle \underbrace{\int_{-L}^{L} \frac{-x\, dx}{(x^2+z^2)^{3/2}}}_{\text{ZERO, Notice integrand is odd}}, \underbrace{\int_{-L}^{L} \frac{z\, dx}{(x^2+z^2)^{3/2}}}_{\text{use trig. substitution}} \right\rangle$$

$$\int_{-L}^{L} \frac{z\, dx}{(x^2+z^2)^{3/2}} = \int_{\theta_1}^{\theta_2} \frac{z^2 \sec^2\theta\, d\theta}{(z^2 \sec^2\theta)^{3/2}}$$

$x = z\tan\theta$	$L = z\tan\theta_2$
$x^2 + z^2 = z^2\sec^2\theta$	$-L = z\tan\theta_1$
$dx = z\sec^2\theta\, d\theta$	$\theta_2 = -\theta_1$

$$= \int_{\theta_1}^{\theta_2} \frac{1}{z} \cos\theta\, d\theta$$

$$= \frac{1}{z}\left(\sin\theta_2 - \sin\theta_1\right)$$



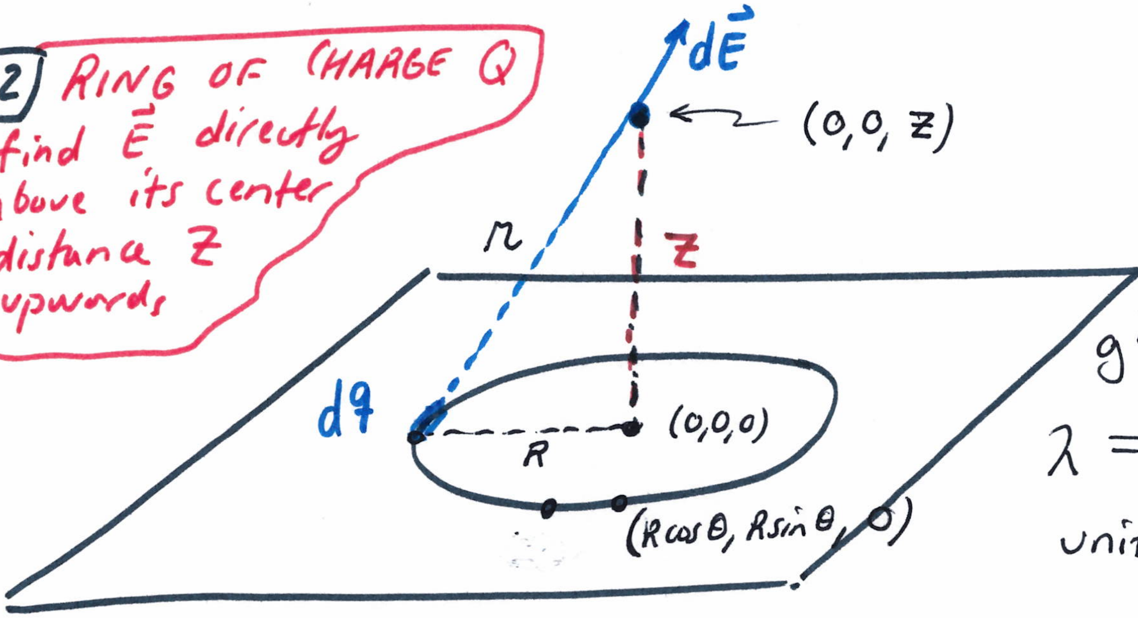
$\sin\theta = \frac{L}{\sqrt{z^2+L^2}}$

$$= \frac{2}{z}\left(\sin\theta_2\right) = \frac{2L}{z\sqrt{z^2+L^2}}$$

$$\therefore \quad \boxed{\vec{E}(0,z) = \left(\frac{2L\lambda}{4\pi\varepsilon_0 z\sqrt{z^2+L^2}}\right)\hat{z}} \xrightarrow{\text{as } z \gg L} \frac{2L\lambda\hat{z}}{4\pi\varepsilon_0 z^2} = \frac{Q\hat{z}}{4\pi\varepsilon_0 z^2}$$

E2 RING OF CHARGE Q

find $\vec{E}$ directly above its center distance z upwards



given,

$$\lambda = \frac{Q}{2\pi R}$$

uniform

$\imath = \sqrt{R^2 + z^2}$ for each point on the circle of charge

$$dq = \lambda\, dl = \lambda R\, d\theta$$

$$\vec{\imath} = \vec{r} - \vec{r}\,' = \langle 0,0,z\rangle - \langle R\cos\theta, R\sin\theta, 0\rangle$$

$$\vec{\imath} = \langle -R\cos\theta, -R\sin\theta, z\rangle$$

$$d\vec{E} = \frac{dq}{4\pi\varepsilon_0 \imath^3}\vec{\imath} = \frac{\lambda R\, d\theta}{4\pi\varepsilon_0 (R^2+z^2)^{3/2}}\langle -R\cos\theta, -R\sin\theta, z\rangle$$

$$\vec{E}(0,0,z) = \frac{\lambda R}{4\pi\varepsilon_0}\int_0^{2\pi} d\theta \Big\langle \frac{-R\cos\theta}{R}$$

$$\vec{E}(0,0,z) = \frac{\lambda R}{4\pi\varepsilon_0 (R^2+z^2)^{3/2}}\int_0^{2\pi} d\theta \langle \underbrace{-R\cos\theta, -R\sin\theta}_{\text{integrate to zero}}, \underbrace{z}_{\text{integrates to } 2\pi z}\rangle$$

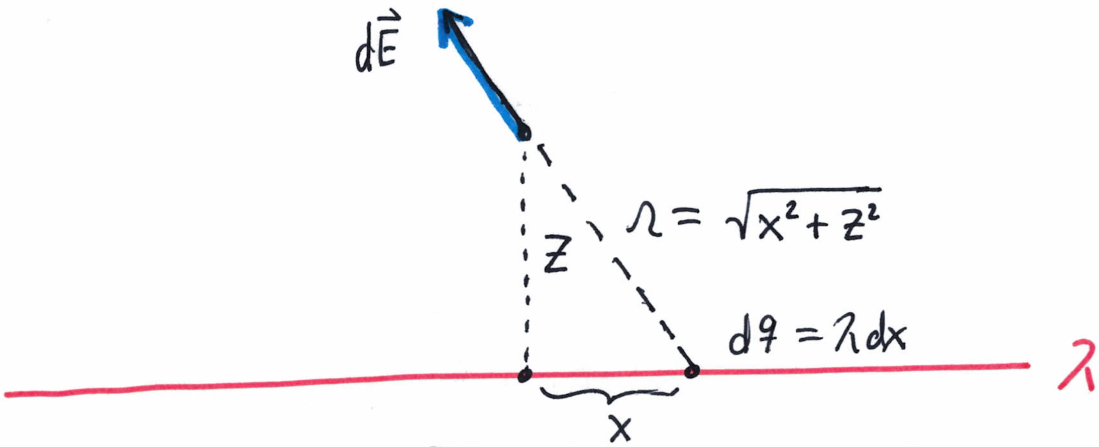
$$\vec{E}(0,0,z) = \frac{2\pi R\lambda z\hat{z}}{4\pi\varepsilon_0 (R^2+z^2)^{3/2}}$$

$Q = 2\pi R\lambda$

$$\boxed{\vec{E}(0,0,z) = \frac{Q}{4\pi\varepsilon_0}\left[\frac{z}{(R^2+z^2)^{3/2}}\right]\hat{z}} \rightarrow \frac{Q\hat{z}}{4\pi\varepsilon_0 z^2}$$

(for $z \gg R$.)

$\boxed{E3}$ Infinite line charge uniform $\lambda = \frac{dq}{dx}$



Same set-up as $\boxed{E1}$ with $L \longrightarrow \infty$ then

$$\int_{-\infty}^{\infty} \frac{z\,dx}{(x^2+z^2)^{3/2}} = \int_{-\pi/2}^{\pi/2} \frac{1}{z}\cos\theta\, d\theta \longleftarrow \boxed{\begin{array}{l} x = z\tan\theta \\ \tan\theta = \frac{x}{z} \\ x \to \pm\infty \\ \Rightarrow \theta \to \pm\frac{\pi}{2} \end{array}}$$

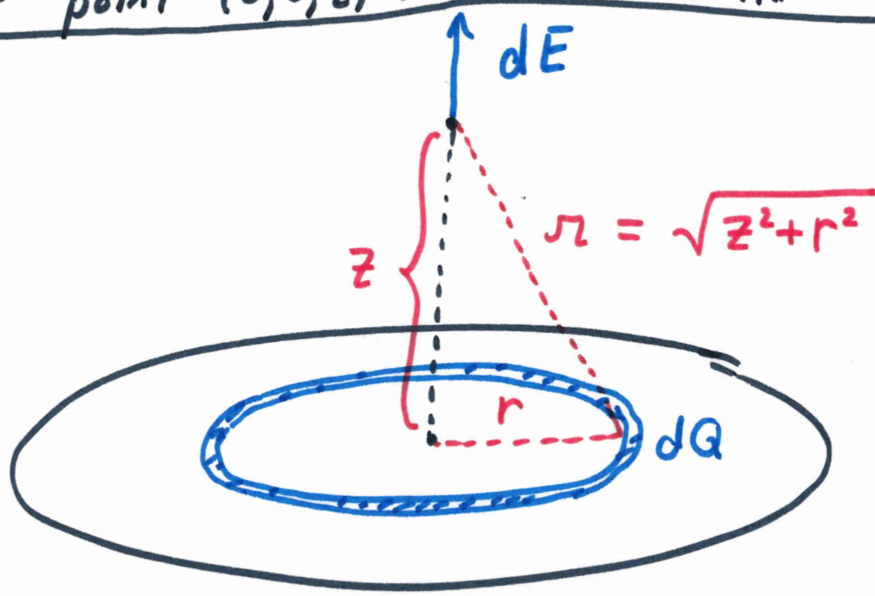
$$= \frac{-\sin\theta}{z}\Bigg|_{-\pi/2}^{\pi/2}$$

$$= \frac{2}{z}$$

$$\therefore \quad \boxed{\vec{E}(0,z) = \frac{2\lambda\hat{z}}{4\pi\varepsilon_0 z} = \left(\frac{\lambda}{2\pi\varepsilon_0}\right)\frac{\hat{z}}{z}}$$

Remark: the infinite line charge does not reduce to the Coulomb field, unlike $\boxed{E1}$.

$\boxed{E4}$ Consider uniform $\sigma = \frac{dq}{da} = \frac{Q}{\pi R^2}$ find $\vec{E}$ at point $(0,0,z)$ above the center of disk at origin.



$$dQ = \sigma\, dA = \sigma(2\pi r\, dr) \quad \text{for } 0 \le r \le R$$

$$d\vec{E} = \frac{dQ}{4\pi\varepsilon_0}\left[\frac{z}{(r^2+z^2)^{3/2}}\right]\hat{z} \quad \text{by } \boxed{E2}$$

$$d\vec{E} = \left[\frac{2\pi\sigma z\, r\, dr}{4\pi\varepsilon_0 (r^2+z^2)^{3/2}}\right]\hat{z}$$

Integrate!

$$\vec{E} = \frac{\sigma z}{2\varepsilon_0}\left[\int_0^R \frac{r\,dr}{(r^2+z^2)^{3/2}}\right]\hat{z}$$

$u = r^2 + z^2$
$du = 2r\,dr$
$r\,dr = \frac{du}{2}$

$$= \frac{\sigma z}{2\varepsilon_0}\left[\int_{z^2}^{R^2+z^2} \frac{du}{2u^{3/2}}\right]\hat{z}$$

$$= \frac{\sigma z}{2\varepsilon_0}\left[\frac{-1}{\sqrt{u}}\Bigg|_{z^2}^{z^2+R^2}\right]\hat{z}$$

$$= \boxed{\frac{\sigma z}{2\varepsilon_0}\left[\frac{1}{z} - \frac{1}{\sqrt{z^2+R^2}}\right]\hat{z}}$$

$\boxed{E4}$ continued / want to consider $z \gg R$,

$$\vec{E}(0,0,z) = \frac{\sigma z}{2\varepsilon_0}\left[\frac{\sqrt{z^2+R^2} - z}{z\sqrt{z^2+R^2}}\right]\hat{z}$$

$$= \frac{\sigma}{2\varepsilon_0}\left[\frac{\sqrt{z^2+R^2} - z}{\sqrt{z^2+R^2}}\right]\hat{z}$$

$$= \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{z}{\sqrt{z^2+R^2}}\right]\hat{z}$$

$$= \frac{\sigma}{2\varepsilon_0}\left[1 - z\left(z^2+R^2\right)^{-1/2}\right]\hat{z}$$

$$= \frac{\sigma}{2\varepsilon_0}\left[1 - \underbrace{z\left(z^2\right.}_{\text{cancel}}\left.\left(1+(R/z)^2\right)\right)^{-1/2}\right]\hat{z}$$

$$= \frac{\sigma}{2\varepsilon_0}\left[1 - \left(1+(R/z)^2\right)^{-1/2}\right]\hat{z}$$

$$= \frac{\sigma}{2\varepsilon_0}\left[1 - \left(1 - \frac{1}{2}\left(\frac{R}{z}\right)^2 + \cdots\right)\right]\hat{z}$$

$$= \frac{\sigma}{2\varepsilon_0}\left(\frac{R^2}{2z^2} + \cdots\right)\hat{z}$$

$$= \frac{Q}{2\pi R^2\varepsilon_0}\frac{R^2}{2z^2}\hat{z} + \cdots = \underbrace{\left(\frac{Q}{4\pi\varepsilon_0}\right)\frac{\hat{z}}{z^2}}_{\text{Coulomb field for } z \gg R} + \cdots$$

Remark: $\vec{E}(0,0,z) \longrightarrow \frac{\sigma}{2\varepsilon_0}\hat{z}$
As $R \longrightarrow \infty$ if we hold z finite.

(infinite planar charge)

Field Lines and Electric Flux

Consider a finite collection of charges $q_1, q_2, \dots, q_n$ at source points $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n$ the electric field at field point $\vec{r}$ is

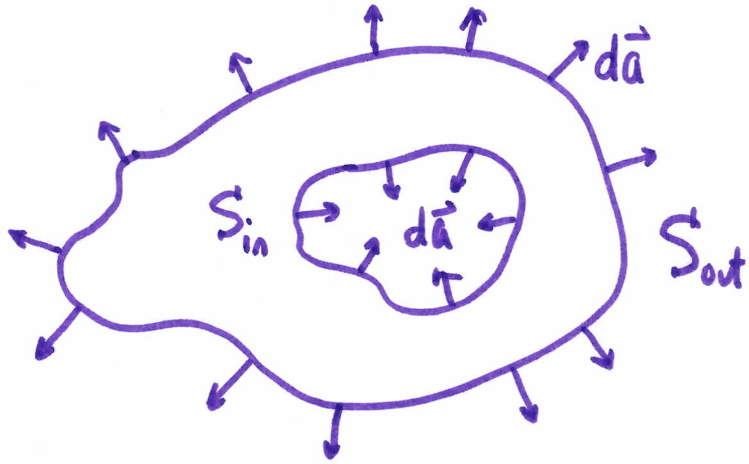
$$\vec{E} = \sum_{i=1}^{n} \frac{q_i}{4\pi\epsilon_0} \frac{\hat{\mathscr{r}}_i}{\mathscr{r}_i^2} \quad \left(\text{for } \vec{\mathscr{r}}_i = \vec{r} - \vec{r}_i\right)$$

Calculate the __divergence__ of this field,

$$\nabla \cdot \vec{E} = \sum_{i=1}^{n} \frac{q_i}{4\pi\epsilon_0} \nabla \cdot \left(\frac{\hat{\mathscr{r}}_i}{\mathscr{r}_i^2}\right) = \sum_{i=1}^{n} \frac{q_i}{4\pi\epsilon_0} \cdot 4\pi\delta^3(\vec{r} - \vec{r}_i)$$

$$\therefore \quad \nabla \cdot \vec{E} = \begin{cases} \infty & \text{for } \vec{r} = \vec{r}_i \\ 0 & \text{for } \vec{r} \neq \vec{r}_i \end{cases}$$

The deformation Th$^{\underline{m}}$ for flux integrals follows from the divergence Th$^{\underline{m}}$ for a solid $\mathcal{V}$ with a hole $\mathcal{V}_{in}$. In such a case $\partial\mathcal{V} = S_{out} - S_{in}$



$d\vec{a}$ for $\partial\mathcal{V}$ always points out of $\mathcal{V}$

- S_{in}, S_{out} both have outward pointing normals
- $-S_{in}$ has inward normal

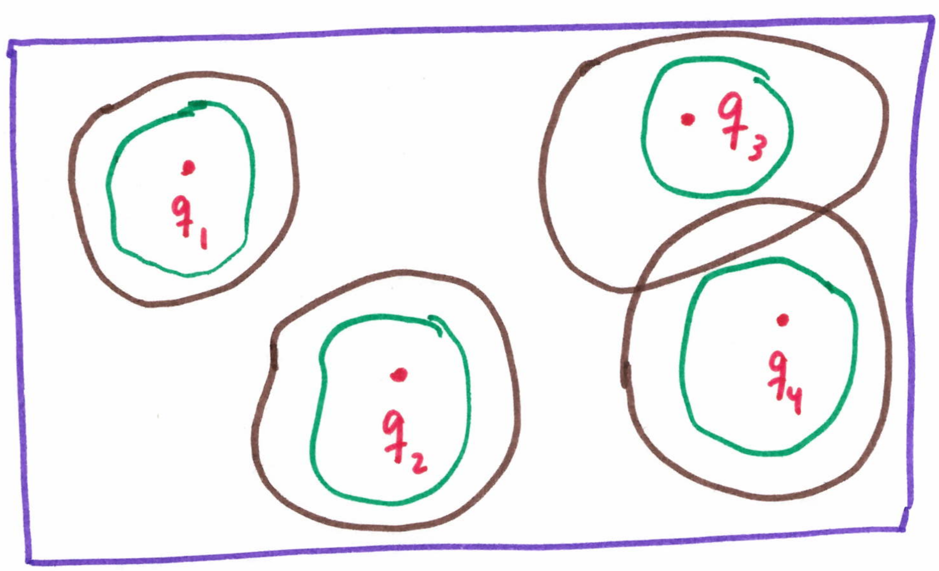
$$\int_{\mathcal{V}} (\nabla \cdot \vec{E})\, d\tau = \int_{\partial \mathcal{V}} \vec{E} \cdot d\vec{a}$$

when $\nabla \cdot \vec{E} = 0$ between S_{in} and S_{out} we obtain $\nabla \cdot \vec{E} = 0$ throughout $\mathcal{V}$ and

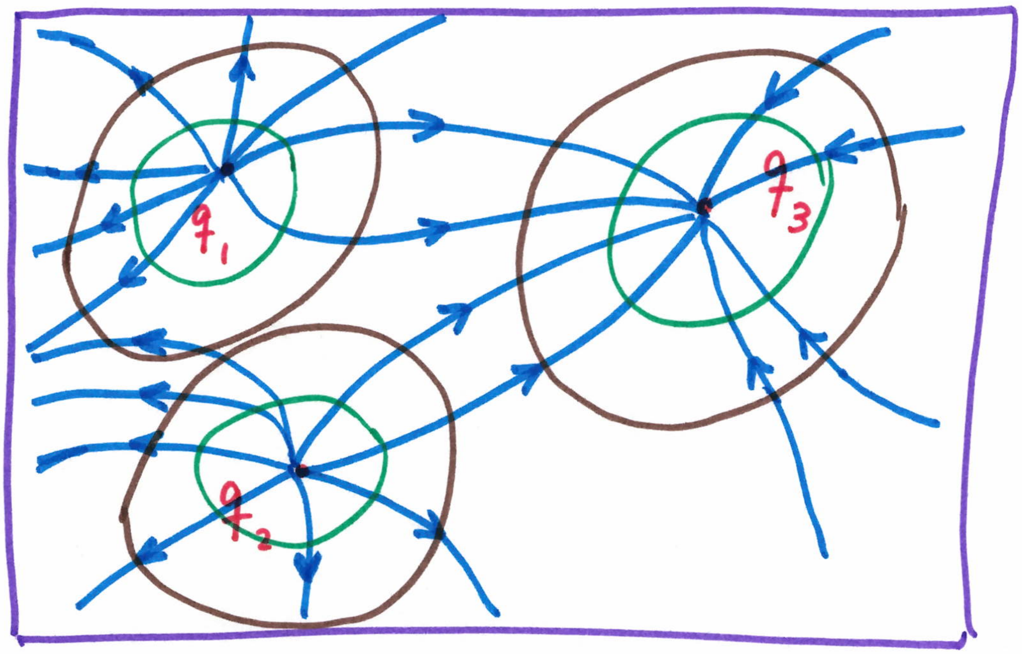
$$0 = \int_{\partial \mathcal{V}} \vec{E} \cdot d\vec{a} = \int_{S_{out}} \vec{E} \cdot d\vec{a} - \int_{S_{in}} \vec{E} \cdot d\vec{a}$$

$$\boxed{\text{Th}^{\underline{m}}/ \quad \int_{S_1} \vec{E} \cdot d\vec{a} = \int_{S_2} \vec{E} \cdot d\vec{a} \quad \text{if } \nabla \cdot \vec{E} = 0 \text{ between the closed surfaces } S_1 \text{ and } S_2}$$

But, the Coulomb Field is just such a field provided we stay away from source charges. The pictured $\underline{S_1}$ and $\underline{S_2}$ below have same flux



It follows we can visualize flux from electric charge by field lines which begin or end at charges



$q_1 = q_2 = Q$

$q_3 = -Q$

# of field lines indicates flux through surface and consequently the charge enclosed.

Remark: field line sketch approximate, the actual pattern must have both $\nabla \cdot \vec{E} = 0$ and $\underbrace{\nabla \times \vec{E} = 0}_{\text{next lecture.}}$ away from charges

DERIVATION OF GAUSS' LAW FROM COULOMB'S

If $\vec{E} = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int_{\mathcal{V}} \rho(\vec{r}\,')\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau$ then

$\nabla\cdot\vec{E} = \rho/\varepsilon_0$ and $\Phi_E = \displaystyle\oint_{\partial\mathcal{V}} \vec{E}\cdot d\vec{a} = \dfrac{Q_{enc}}{\varepsilon_0}$

The key identity is $\nabla\cdot\left(\dfrac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\delta^{(3)}(\vec{r}-\vec{r}\,')$ given $\vec{\mathscr{r}} = \vec{r}-\vec{r}\,'$ as usual, CALCULATE,

$$\nabla\cdot\vec{E} = \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}} \rho(\vec{r}\,')\,\nabla\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)d\tau$$

$$= \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}} \rho(\vec{r}\,')\,4\pi\delta^{(3)}(\vec{r}-\vec{r}\,')\,d\tau$$

$$= \frac{1}{\varepsilon_0}\rho(\vec{r}) \quad \therefore \boxed{\nabla\cdot\vec{E} = \rho/\epsilon_0}$$

differential form of Gauss' Law.

Thus,

$$\Phi_E = \int_{\partial\mathcal{V}} \vec{E}\cdot d\vec{a}$$

$$= \int_{\mathcal{V}} (\nabla\cdot\vec{E})\,d\tau = \int_{\mathcal{V}} \frac{\rho(\vec{r})}{\epsilon_0}\,d\tau = \frac{1}{\varepsilon_0}\underbrace{\int_{\mathcal{V}} \rho(\vec{r})\,d\tau}_{Q_{enc}\text{ within }\mathcal{V}}$$

$$\therefore \boxed{\Phi_E = \frac{Q_{enc}}{\varepsilon_0}}$$