LECTURE 4: DEFINITION OF TOPOLOGY

- Read the introduction to Chapter 3 of Manetti, pg. 39.

Def^{n}/ A <u>topology</u> on a set X is a family $\mathcal{T}$ of subsets of X called <u>open</u> <u>sets</u> satisfying the following requirements

elements of topological space are typically called <u>points</u>

A1.) $\emptyset, X \in \mathcal{T}$ (both $\emptyset$ and X are open sets)

A2.) union of open sets is open

A3.) intersection of open sets A, B is open

X paired with $\mathcal{T}$ forms a topological space.

Th^{m}/ intersection of finitely many open sets is open in a topological space.

<u>Proof</u>: Let $A_1, \dots, A_n \in \mathcal{T}$ for some $n \in \mathbb{N}$. Notice $A_1 \cap A_2 \in \mathcal{T}$ by A3. $A_1 \cap A_2 \cap A_3 = (A_1 \cap A_2) \cap A_3$ thus $A_1 \cap A_2 \cap A_3 \in \mathcal{T}$ by A3. Inductively suppose $A_1 \cap \dots \cap A_{n-1} \in \mathcal{T}$ and consider $A_1 \cap \dots \cap A_{n-1} \cap A_n = (A_1 \cap \dots \cap A_{n-1}) \cap A_n$ thus $A_1 \cap \dots \cap A_n \in \mathcal{T}$ by A3 and the induct. hypo. //

$\boxed{E1}$ Let X be a set then $\mathcal{T} = \{\emptyset, X\}$ defines a topology called the <u>TRIVIAL</u> topology

$\boxed{E2}$ Let X be a set then $\mathcal{T} = \mathcal{P}(X)$ defines the <u>DISCRETE</u> topology. Literally every subset of X is open in the discrete topology.

<u>Remark</u>: $\boxed{E1}$ and $\boxed{E2}$ show there are more than one topologies for any $X \neq \emptyset$.

$\boxed{E2}$ Let $X = \{a, b, c\}$ then $\mathcal{T} = \{\emptyset, X, \{a,b\}, \{b\}, \{b,c\}\}$

then $\{a,b\} \cup \{b,c\} = \{a,b,c\}$ and $\{a,b\} \cup \{b\} = \{a,b\}$

and $\{a,b\} \cap \{b\} = \{b\}$ and $\{a,b\} \cap \{b,c\} = \{b\}$ etc..

We may verify $\mathcal{T}$ is topology on X.

$$\mathcal{P}(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}\}$$

notice $\mathcal{T} \subset \mathcal{P}(X)$ hence $\mathcal{T}$ is <u>coarser</u> than $\mathcal{P}(X)$
or $\mathcal{P}(X)$ is <u>finer</u> than $\mathcal{T}$

language for comparing topologies $\mathcal{T}_1$, $\mathcal{T}_2$ on X provided either $\mathcal{T}_1 \subset \mathcal{T}_2$ or $\mathcal{T}_2 \subset \mathcal{T}_1$

$\boxed{E3}$ <u>Euclidean topology on $\mathbb{R}$</u>

$U \in \mathcal{T}$ iff U = union of open intervals of $\mathbb{R}$ or $U = \emptyset$

Since $\emptyset, (-\infty, \infty) = \mathbb{R} \in \mathcal{T}$ we satisfy A1 and A2 is true since arbitrary unions of unions is once more a union.

Let $A = \bigcup_I A_i$ and $B = \bigcup_J B_j$ where A_i, B_j are open intervals $\forall I, J$

then
$$A \cap B = \left(\bigcup_I A_i\right) \cap \left(\bigcup_J B_j\right) = \bigcup_{I,J} \underbrace{A_i \cap B_j} \in \mathcal{T}$$

intersection of any two open intervals is either $\emptyset$ or an open interval once more.

Open Intervals : (a,b), (a,∞), $(-\infty, a)$, $(-\infty, \infty)$

(3)

$\boxed{E4}$ $\mathcal{T} = \{\emptyset\} \cup \{(-\infty, a) \mid a \in \mathbb{R} \cup \{+\infty\}\}$ gives <u>upper</u> <u>topology</u> for $\mathbb{R}$

$\emptyset$, $(-\infty, \infty) = \mathbb{R} \in \mathcal{T}$ so A1 ✓

<u>A2</u>: $\bigcup_{i\in I} (-\infty, a_i) = (-\infty, a)$ where $a = \sup\{a_i \mid i \in I\}$ or $a = \infty$

<u>A3</u>: $\left(\bigcup_{i\in I} (-\infty, a_i)\right) \cap \left(\bigcup_{j\in J} (-\infty, b_j)\right) = \bigcup_{\substack{i\in I\\ j\in J}} (-\infty, a_i) \cap (-\infty, b_j) = (-\infty, m)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \sup \{\min(a_i, b_j) \mid i \in I, j \in J\}$

where $m = \cancel{\inf} \{a_i, b_j \mid i \in I, j \in J\}$

or $m = \infty$.

Def^n/ Let $X, \mathcal{T}$ be topological space. Then $C \subseteq X$ is called <u>closed</u> if $X - C$ is open.

Th^m/
(C1) $\emptyset$, X are closed
(C2) arbitrary intersection of closed sets is closed
(C3) union of two closed sets is closed

could define topology by declaring which sets are closed.

$\boxed{E5}$ Cofinite topology: $C \subseteq X$ closed iff $C = X$ or C is finite.
(then open sets are those which complement C)

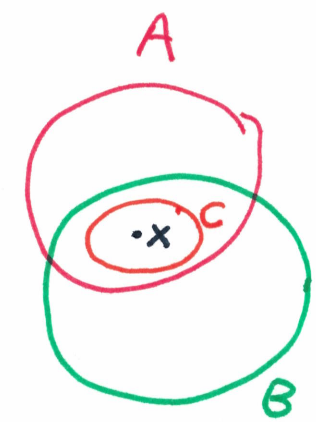
Def$^{\underline{n}}$/ A topological space $X, \mathcal{T}$ has basis $\mathcal{B}$ if $\mathcal{B} \subset \mathcal{T}$ and every $\phi \neq A \in \mathcal{T}$ can be expressed as union of sets in $\mathcal{B}$.

$\boxed{E6}$ for $\boxed{E3}$ the set of all open intervals forms a basis for Euclidean Top. on $\mathbb{R}$. Likewise $\mathcal{B} = \{(-\infty, a) \mid a \in \mathbb{R} \cup \{\infty\}\}$ forms basis for the upper topology discussed in $\boxed{E4}$.

Th$^{\underline{m}}$ (3.7)/ Let X be a set and $\mathcal{B} \subset \mathcal{P}(X)$. There exists a topology on X for which $\mathcal{B}$ is a basis iff the following two conditions hold:

1.) $X = \bigcup \{B \mid B \in \mathcal{B}\}$

2.) for any pair $A, B \in \mathcal{B}$ and any point $x \in A \cap B$ there exists $C \in \mathcal{B}$ such that $x \in C \subset A \cap B$



Proof:

$\Rightarrow$/ Let X be a set and $\mathcal{B} \subset \mathcal{P}(X)$ which defines basis for topology $\mathcal{T}$ meaning every open nonempty set can be expressed as union of sets in $\mathcal{B}$. Since $X \in \mathcal{T}$ it follows $\exists$ collection of sets in $\mathcal{B}$ whose union is X. Then taking the union over all $B \in \mathcal{B}$ must likewise produce X. Since $A \cup A = A$ we find $A \in \mathcal{B} \Rightarrow A \in \mathcal{T}$. If $A, B \in \mathcal{B}$ then $A, B \in \mathcal{T}$ thus $A \cap B \in \mathcal{T}$. If $x \in A \cap B = \underbrace{\bigcup_I B_I}_{B_I \in \mathcal{B} \text{ for each } I}$ then $\exists\, i_0 \in I$ for which $x \in B_{i_0}$ and $C = B_{i_0} \subseteq A \cap B$ with $C \in \mathcal{B}$

$\therefore$ 1.) and 2.) hold true.

Proof Continued

$\Leftarrow$) Suppose X is set and $\mathcal{B} \subset \mathcal{P}(X)$ for which 1.) and 2.) hold true that is $X = \bigcup\{B \mid B \in \mathcal{B}\}$ and for any pair $A, B \in \mathcal{B}$ and $x \in A \cap B$ $\exists\, C \in \mathcal{B}$ such that $x \in C \subseteq A \cap B$. We wish to show $\mathcal{B}$ is basis for a topology $\mathcal{T}$ on X. Claim: we may define a topology $\mathcal{T}$ on X by generating its open sets as unions of sets in $\mathcal{B}$ and the empty set.

A1) Clearly both $X, \phi \in \mathcal{T}$ since they're formed by unions of sets from $\mathcal{B}$ by 1. and $\phi \in \mathcal{T}$ by definition. (Manetti cites the "empty union" which I forget at times)

A2) If $A_i \in \mathcal{T}$ for $i \in I$ then $\exists\, j_i \in J_i$ for which $A_i = \bigcup_{j_i \in J_i} B_{j_i}$ where $B_{j_i} \in \mathcal{B}$ then $\bigcup_{i \in I} A_i = \bigcup_{i \in I} \bigcup_{j_i \in J_i} B_{j_i} \in \mathcal{T}$ since this is a union over $\mathcal{B}$

A3) Let $A, B \in \mathcal{T}$ then $A = \bigcup_I B_I$ and $B = \bigcup_J B_J$ where $A_I, B_J \in \mathcal{B}$

hence $A \cap B = \bigcup_{I,J} B_I \cap B_J$.

and $B_I \cap B_J = \bigcup\{C \mid C \in \mathcal{B},\ C \subset B_I \cap B_J\}$ by assumption 2.)

thus $A \cap B \in \mathcal{T}$ as it is formed by union of $B_I \cap B_J \in \mathcal{B}$. //

E7 <u>LOWER-LIMIT LINE</u> $\mathcal{T}_{LL}$ • See (footnote) comment on pg. 41 of Manetti:

Topology for $\mathbb{R}$ generated by basis consisting of $[a,b)$.

Since $(a,b) = \bigcup_{c>a} [c,b)$ we find $(a,b) \in \mathcal{T}_{LL}$ but $[a,b) \notin \mathcal{T}_{\text{Euclidean}}$

The LOWER-LIMIT top. on $\mathbb{R}$ is <u>finer</u> than the Euclidean top. on $\mathbb{R}$.

E8 <u>DISJOINT UNION</u> ($\mathbb{R} \sqcup \mathbb{R} \neq \mathbb{R}$, the disjoint union is much like a CARTESIAN PRODUCT)

Let X_i for $i \in I$ be topological spaces and form the <u>disjoint</u> <u>union</u> $X = \bigsqcup_{i \in I} X_i$ can be given a topology (most coarse) where $A \subseteq X$ is open iff $A \cap X_i$ is open $\forall i \in I$.

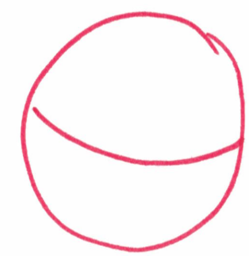
<u>Remark</u>: §3.4 on metric spaces brings many further examples which you might find familiar. The point of this section is just to give some less familiar examples. What follows next is quite divorced from the usual

$\boxed{\text{E10}}$ $\mathbb{K} = \mathbb{R}$ look at Z.T. on $\mathbb{R}^3$

$$f(x,y,z) = x^2+y^2+z^2-1$$

$$D(f) = \{(x,y,z) \mid x^2+y^2+z^2-1 \neq 0\}$$

$$V(f) = \{(x,y,z) \mid x^2+y^2+z^2 = 1\} \leftarrow \text{unit-sphere}$$



is closed in Z.T.

E9 ZARISKI TOPOLOGY

Let $\mathbb{K}$ be a field and $n \in \mathbb{N}$ and $\mathbb{K}[x_1, \dots, x_n]$ the polynomial ring in n-variables with coefficients in $\mathbb{K}$. For $f \in \mathbb{K}[x_1, \dots, x_n]$ define $D(f) = \{(a_1, \dots, a_n) \in \mathbb{K}^n \mid f(a_1, \dots, a_n) \neq 0\}$. Since $D(0) = \emptyset$ and $D(1) = \mathbb{K}^n$ and $D(f) \cap D(g) = D(fg)$ we may use the set of $D(f)$ to generate a topology on $\mathbb{K}^n$ called <u>Zariski Topology</u>.

Let $V(f) = \mathbb{K}^n - D(f) = \{(a_1, \dots, a_n) \in \mathbb{K}^n \mid f(a_1, \dots, a_n) = 0\}$ and

$$V(E) = \bigcap_{f \in E} V(f) = \{a \in \mathbb{K}^n \mid f(a) = 0 \ \forall f \in E\}$$

for any $E \subseteq \mathbb{K}^n[x_1, \dots, x_n]$. You can argue $V(E)$ are only closed sets in Z.T.

If $E \subseteq \mathbb{K}^n$ then $I = (E)$ is the <u>ideal</u> generated by E.
As $E \subseteq I$ it follows $V(E) \subseteq V(E)$. Also, if $f \in I$ then $\exists n \in \mathbb{N}$ and polynomials $f_1, \dots, f_n \in E$ and $g_1, \dots, g_n \in \mathbb{K}[x_1, \dots, x_n]$ such that $f = f_1 g_1 + \dots + f_n g_n$ hence $V(E) \subset V(f_1) \cap \dots \cap V(f_n) \subset V(f)$
Thus $V(E) \subset \bigcap_{f \in I} V(f) = V(I)$ $\therefore$ $V(I) = V(E)$.
$\Rightarrow$ Zariski closed sets are those of type $V(I)$ as I is ideal of $\mathbb{K}[x_1, \dots, x_n]$

(Will need to consult D&F for details.)