Lecture 5: Gauss' Law & Dirac Deltas

In differential form, if the charge density $\rho = \frac{dQ}{d\tau}$ then Gauss' Law is

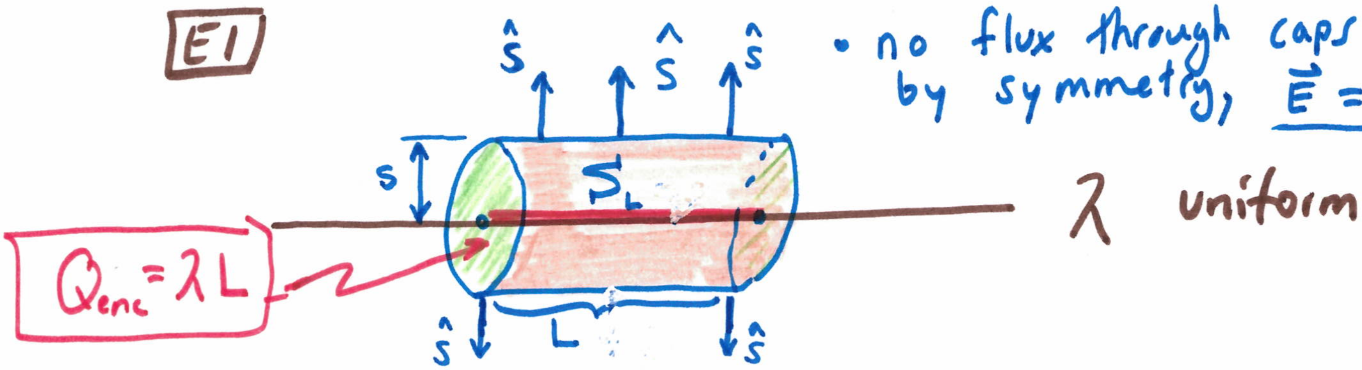
$$\boxed{\nabla \cdot \vec{E} = \rho/\varepsilon_0}$$

If we integrate the above over $\mathcal{V}$ where $\mathcal{V}$ is a simple solid (no holes) then the flux Φ_E through $\partial\mathcal{V}$ has

$$\Phi_E = \int_{\partial\mathcal{V}} \vec{E} \cdot d\vec{a} = \int_{\mathcal{V}} (\nabla \cdot \vec{E})\, d\tau = \int \frac{\rho}{\varepsilon_0} d\tau = \frac{1}{\varepsilon_0}\int \rho\, d\tau$$

$$\therefore \quad \boxed{\Phi_E = \frac{1}{\varepsilon_0}\int_{\mathcal{V}} \rho\, d\tau = \frac{Q_{enc}}{\varepsilon_0}}$$

E1



- no flux through caps by symmetry, $\underline{\vec{E} = E\hat{s}}$.

$$\Phi_E = \int_{\partial\mathcal{V}} \vec{E} \cdot d\vec{a} = \int_{S} (E\hat{s}) \cdot (\hat{s}\, da) = E \int_{S} da$$

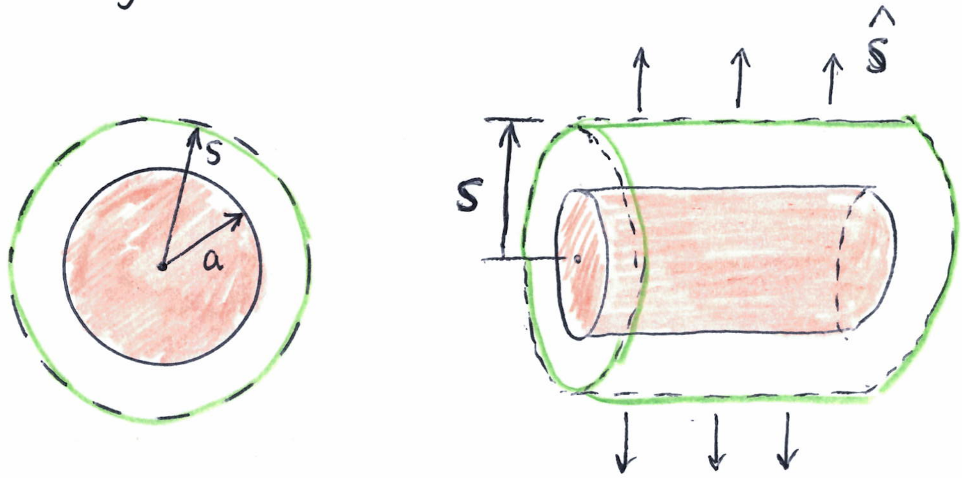
$$\Phi_E = \frac{Q_{enc}}{\varepsilon_0} \Rightarrow E(2\pi s L) = \frac{\lambda L}{\varepsilon_0}$$

$$\therefore \quad \boxed{\vec{E} = \frac{\lambda}{2\pi\varepsilon_0}\frac{\hat{s}}{s}}$$

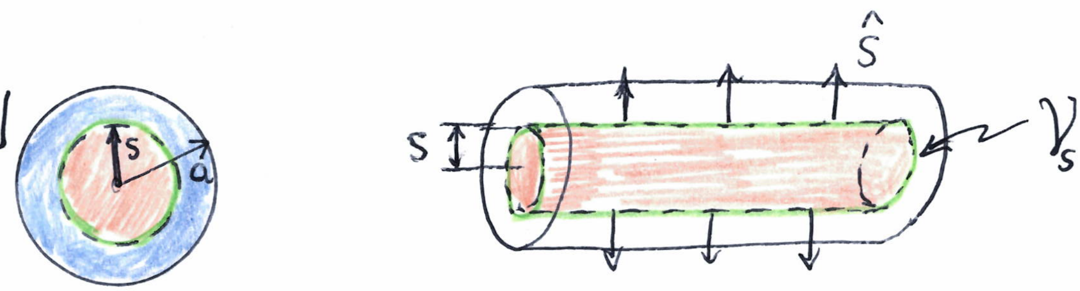
E2 consider cylindrical charge with uniform density $\rho = \frac{dQ}{d\tau}$ for $0 \leq s \leq a$

Consider length L subset of the cylinder,

Ⓘ $s \geq a$



Ⓘⓘ $0 \leq s \leq a$



Both Ⓘ and Ⓘⓘ [the cylinder is infinite in length so we see $\vec{E}$ must point $\perp$ to axis by symmetry; $\vec{E} = E\hat{s}$]

Ⓘ. $\frac{Q_{enc}}{\varepsilon_0} = \Phi_E \Rightarrow \frac{\pi a^2 L \rho}{\varepsilon_0} = E(2\pi s L)$

$$\therefore \boxed{\vec{E} = \frac{a^2 \rho}{2\varepsilon_0 s}\hat{s} = \frac{\lambda}{2\pi\varepsilon_0}\frac{\hat{s}}{s} \quad \text{for } s \geq a \text{ where}}$$

$$\lambda = \pi a^2 \rho = \frac{dQ}{dz}$$

if we use z as coordinate of axis.

E2 continued

II $\quad 0 \leq s \leq a$

$$Q_{enc} = \int_{\mathcal{V}_s} \rho \, d\tau = \int_0^{2\pi} \int_0^{s} \int_0^{L} \rho \cdot s \, dz \, d\tilde{s} \, d\phi$$

$$Q_{enc} = \rho \cdot \underbrace{\pi s^2 L}$$

volume of $\mathcal{V}_s$ where

$0 \leq z \leq L$ and $0 \leq \tilde{s} \leq s$

$$\frac{Q_{enc}}{\varepsilon_0} = \Phi_E$$

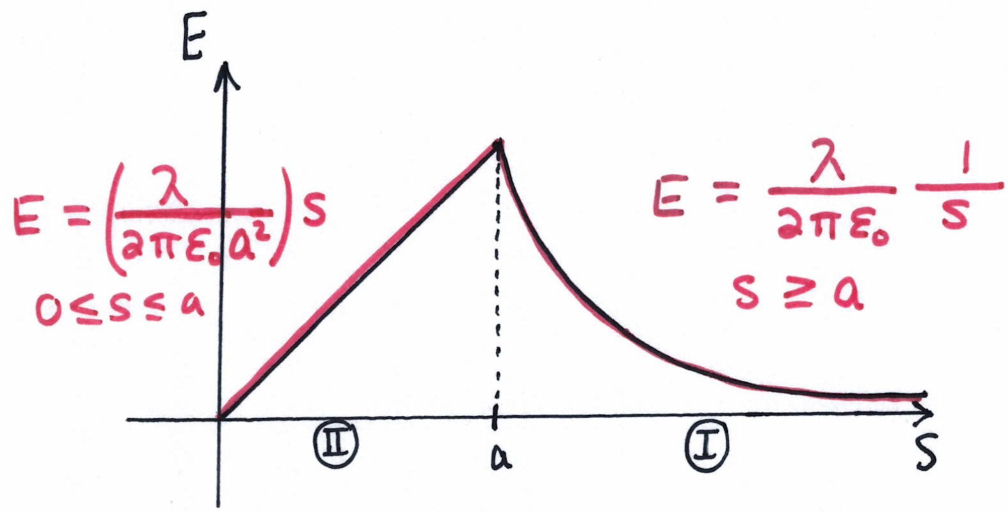
charge per unit length

$$\underbrace{\lambda = \pi a^2 \rho}$$

$$\frac{\rho \pi s^2 L}{\varepsilon_0} = (2\pi s L) E$$

$$E = \frac{\rho \pi s^2 L}{2\pi s L \varepsilon_0} = \left(\frac{\rho}{2\varepsilon_0}\right) s = \left(\frac{\rho \pi a^2}{2\pi \varepsilon_0 a^2}\right) s$$

$$\boxed{\vec{E} = \left(\frac{\lambda}{2\pi \varepsilon_0} \cdot \frac{s}{a^2}\right) \hat{s} \quad \text{for } 0 \leq s \leq a}$$



<u>Remark:</u> there is a significant distinction between $\boxed{E1}$ vs. $\boxed{E2}$

For $\boxed{E2}$ we face no divergence at $s=0$ in fact $\nabla \cdot \vec{E} = 0$ on <u>all</u> of $\mathbb{R}^3$ with $s > a$ and for $0 \leq s \leq a$,

$$\nabla \cdot \vec{E} = \nabla \cdot \left(\frac{\lambda}{2\pi \varepsilon_0 a^2} s \hat{s} \right) = \frac{1}{s} \frac{\partial}{\partial s} \left[s \cdot \frac{\lambda s}{2\pi \varepsilon_0 a^2} \right]$$

$$= \frac{1}{s} \left(\frac{2 s \lambda}{2\pi \varepsilon_0 a^2} \right)$$

$$= \frac{\lambda}{\pi a^2 \varepsilon_0}$$

$$= \frac{\pi a^2 \rho}{\pi a^2 \varepsilon_0}$$

$$= \rho / \varepsilon_0$$

In contrast, for $s > a$,

$$\left.\begin{aligned} \nabla \cdot \vec{E} &= \nabla \cdot \left(\frac{\lambda}{2\pi \varepsilon_0} \frac{1}{s} \right) \\ &= \frac{\lambda}{2\pi \varepsilon_0 s} \frac{\partial}{\partial s} \left[s \cdot \frac{1}{s} \right] \\ &= \frac{\lambda}{2\pi \varepsilon_0 s} \frac{\partial}{\partial s} [1] \\ &= 0 \end{aligned}\right\} \begin{array}{l} s > a \\ \text{so } s \neq 0 \\ \text{for } \boxed{E2} \end{array}$$

HOWEVER for $\boxed{E1}$

For $\boxed{E1}$, $\nabla \cdot \vec{E} = \nabla \cdot \left(\frac{\lambda}{2\pi \varepsilon_0} \frac{\hat{s}}{s} \right) = \frac{\lambda}{2\pi \varepsilon_0} \nabla \cdot \left(\frac{\hat{s}}{s} \right)$

Remark continued:

For $\boxed{E1}$, $\vec{E} = \dfrac{\lambda}{2\pi\varepsilon_0}\dfrac{\hat{s}}{s}$ is divergent at $s=0$. In fact, you can argue,

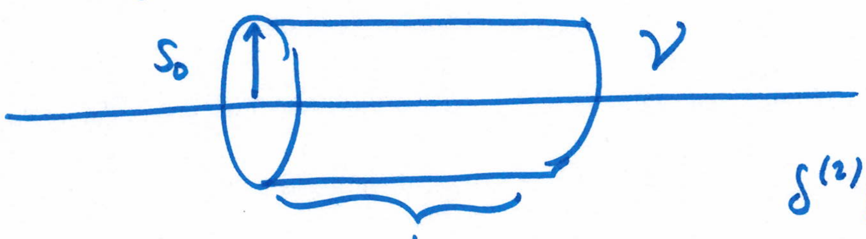
$$\boxed{\nabla\cdot\left(\frac{\hat{s}}{s}\right) = 2\pi\,\delta(s)\,\delta(\phi)}$$

Notice,

$$\nabla\cdot\vec{E} = \frac{\lambda}{2\pi\varepsilon_0}\,\nabla\cdot\left(\frac{\hat{s}}{s}\right) = \frac{\lambda}{2\pi\varepsilon_0}\cdot 2\pi\,\delta^2(\vec{s})$$

$$\Rightarrow \boxed{\nabla\cdot\vec{E} = \frac{\lambda\,\delta^2(\vec{s})}{\varepsilon_0}}$$

Apparently, $\rho = \lambda\,\delta^2(\vec{s})$ for the infinite line charge of constant charge density λ,



$$\delta^{(2)}(\vec{s}) = \delta(s)\,\delta(\phi)$$

$$Q_{enc} = \int_{\mathcal{V}} \rho\, d\tau = \int_0^L \int_0^{2\pi} \int_0^{s_0} \lambda\,\delta^2(\vec{s})\, ds\, d\phi\, dz =$$

$$= \lambda \int_0^L dz$$

$$= \lambda L$$

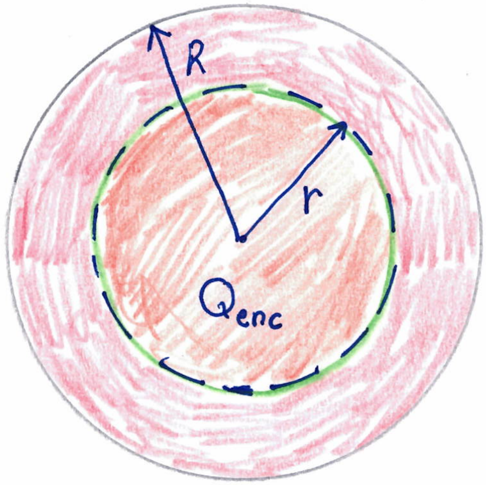
$$\frac{Q_{enc}}{\varepsilon_0} = \Phi_E = 2\pi s L E$$

$$\frac{\lambda L}{\varepsilon_0} = 2\pi s L E$$

$$\therefore\ \underline{E = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{s}}$$

$\boxed{E3}$ Consider <u>uniformly</u> charged sphere of radius R with charge Q. To find E we consider two cases.

I. $0 \le r \le R$



$$Q_{enc} = \left(\frac{4}{3}\pi r^3\right)\rho$$

$$= \left(\frac{4}{3}\pi r^3\right)\left(\frac{Q}{\frac{4}{3}\pi R^3}\right)$$

$$= \frac{r^3 Q}{R^3}$$

The Gaussian surface at radius r has a spherical symmetry and hence $\vec{E} = E\hat{r}$

thus $\Phi_E = \int_{S_r} \vec{E}\cdot d\vec{a} = E\int_{S_r} da = 4\pi r^2 E$

$$\frac{Q_{enc}}{\varepsilon_0} = \Phi_E \Rightarrow \frac{r^3 Q}{\varepsilon_0 R^3} = 4\pi r^2 E$$

$$\therefore \boxed{\vec{E} = \left(\frac{rQ}{4\pi\varepsilon_0 R^3}\right)\hat{r} \quad \text{for } 0 \le r \le R}$$

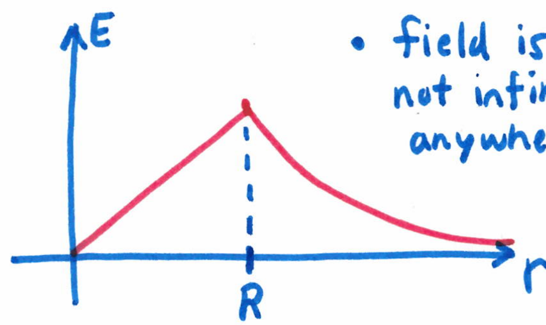
II. $r > R$ then,

$Q_{enc} = Q$ and hence

$$\frac{Q_{enc}}{\varepsilon_0} = \Phi_E$$

$$\frac{Q}{\varepsilon_0} = 4\pi r^2 E$$

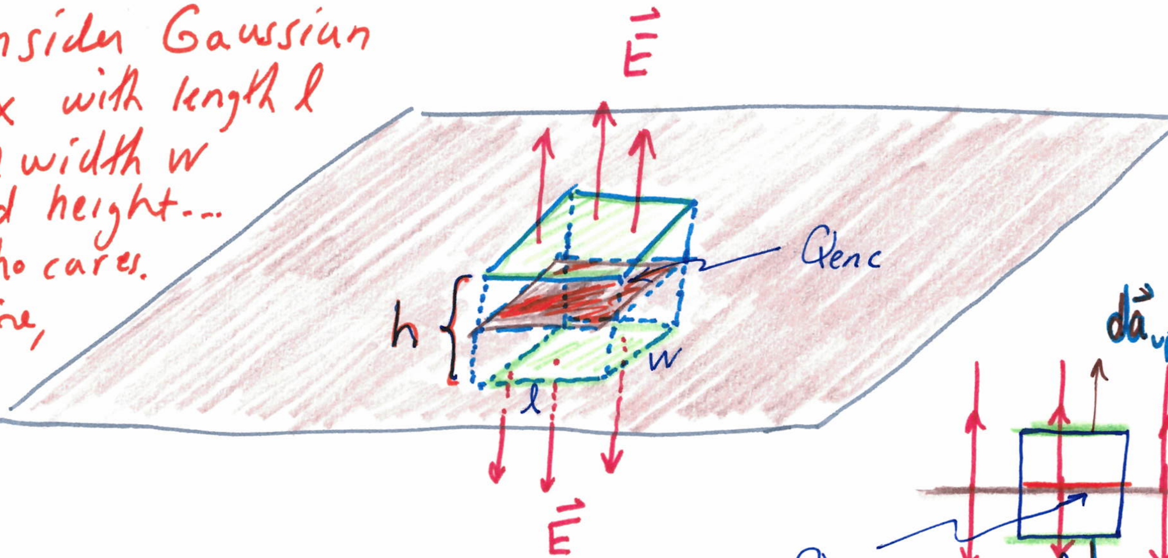
$$\therefore \boxed{\vec{E} = \frac{Q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2} \quad r \ge R}$$

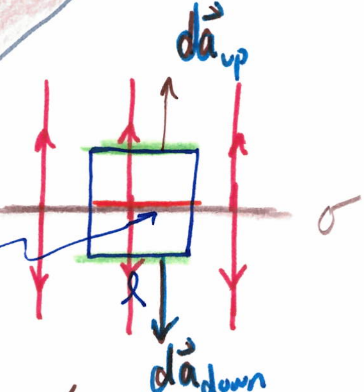


- field is not infinite anywhere.

E4 Consider infinite plane charge of constant charge density σ at $z = 0$.

consider Gaussian Box with length l and width w and height... who cares. Fine, h





$\text{AREA} = lw$, $Q_{enc} = \sigma l w$

plane <u>infinite</u> $\Rightarrow$ E must be normal to plane since otherwise a direction would be preferred violating the symmetry.

$$\therefore \quad \vec{E}_{above} = E\hat{z} \quad , \; d\vec{a}_{up} = (da)\hat{z}$$

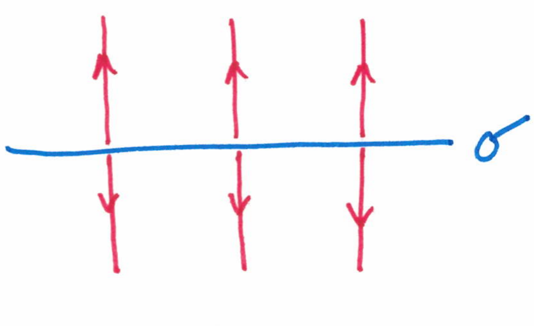
$$\vec{E}_{below} = E(-\hat{z}) \quad d\vec{a}_{down} = (da)(-\hat{z})$$

<u>GAUSS' LAW</u>

$$\frac{Q_{enc}}{\epsilon_0} = \Phi_E$$

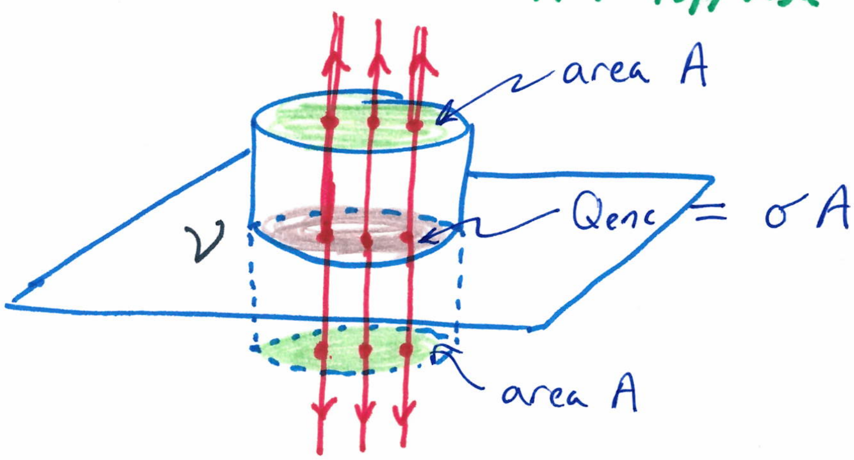
$$\frac{\sigma l w}{\epsilon_0} = \Phi_{up} + \Phi_{down} = Elw + Elw = 2Elw$$

$$\therefore \quad \boxed{\vec{E} = \frac{\sigma}{2\epsilon_0}\frac{|z|}{z}\hat{z} = \begin{cases} \frac{\sigma}{2\epsilon_0}\hat{z} & : z > 0 \\ \frac{-\sigma}{2\epsilon_0}\hat{z} & : z < 0 \end{cases}}$$



$E = \frac{\sigma}{2\epsilon_0}$ for $z \neq 0$

E4 revisited (don't have to use box, really anything with flat top/base works same)



$$\frac{Q_{enc}}{\varepsilon_0} = \Phi_E$$

$$\frac{\sigma A}{\varepsilon_0} = 2AE$$

$$\therefore \ \sigma = E = \frac{\sigma}{2\varepsilon_0}.$$

$$\int_{\mathcal{V}} (\nabla\cdot\vec{E})\,d\tau = \int_{\partial\mathcal{V}} \vec{E}\cdot d\vec{a} = 2AE$$

$$\nabla\cdot\vec{E} = \begin{cases} 0 & \text{if } z \neq 0 \\ \infty & \text{if } z = 0 \end{cases}$$

Comment:

$$\frac{z}{|z|} = \begin{cases} 1 & \text{if } z > 0 \\ -1 & \text{if } z < 0 \end{cases}$$

$$\boxed{\rho(x,y,z) = \sigma\,\delta(z)}$$

$$\int_{\mathcal{V}} (\nabla\cdot\vec{E})\,d\tau = \int_{\mathcal{V}} \frac{\sigma\,\delta(z)}{\varepsilon_0}\,d\tau$$

$$= \iint_A \int_{-h}^{h} \frac{\sigma\,\delta(z)}{\varepsilon_0}\,dz\,dx\,dy$$

$$= \frac{\sigma}{\varepsilon_0}\iint_A dx\,dy$$

$$= \frac{\sigma A}{\varepsilon_0}$$

$$\boxed{\nabla\cdot\left(\frac{z}{2|z|}\right) = \delta(z)}$$

Remark: think about $\frac{d}{dx}\left(\Theta(x)\right) = \delta(x)$

$$\vec{E} = \left(\frac{\sigma}{2\varepsilon_0}\,\frac{z}{|z|}\right)\hat{z} \quad\Rightarrow\quad \nabla\cdot\vec{E} = \frac{\sigma\,\delta(z)}{\varepsilon_0}$$

DIVERGENCE ASSOCIATED TO POINT, LINE and PLANE charges in $\mathbb{R}^3$

$$\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\,\delta^{(3)}(\vec{r})$$

$\vec{r} = \langle x, y, z\rangle$

$\delta^{(3)}(\vec{r}) = \delta(x)\delta(y)\delta(z)$

$$\nabla\cdot\left(\frac{\hat{s}}{s}\right) = 2\pi\,\delta^{(2)}(\vec{s})$$

$\vec{s} = \langle x, y\rangle$

$\delta^{(2)}(\vec{s}) = \delta(x)\delta(y)$

$$\nabla\cdot\left(\frac{z}{\sqrt{z^2}}\right) = \nabla\cdot\left(2\Theta(z)-1\right) = 2\delta(z)$$

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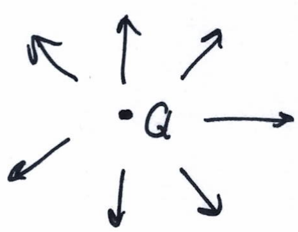
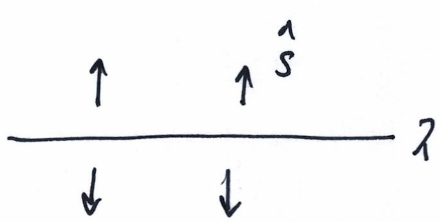
(1) $\vec{E} = \dfrac{Q}{4\pi\varepsilon_0}\dfrac{\hat{r}}{r^2}$

$$\nabla\cdot\vec{E} = \frac{Q}{4\pi\varepsilon_0}\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) = \frac{Q}{4\pi\varepsilon_0}\left(4\pi\,\delta^3(\vec{r})\right)$$



$\nabla\cdot\vec{E} = \rho/\varepsilon_0$ where $\rho = Q\,\delta^3(\vec{r})$

$$\rho = \begin{cases} \infty & \text{at } \vec{r} = 0 \\ 0 & \text{at } \vec{r} \neq 0 \end{cases}$$



$$\int_{\mathcal{V}} Q\,\delta^3(\vec{r}) = \begin{cases} Q & \text{if } 0 \in \mathcal{V} \\ 0 & \text{if } 0 \notin \mathcal{V} \end{cases}$$

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(2) $\vec{E} = \dfrac{\lambda}{2\pi\varepsilon_0}\dfrac{\hat{s}}{s}$

$$\nabla\cdot\vec{E} = \frac{\lambda}{2\pi\varepsilon_0}\nabla\cdot\left(\frac{\hat{s}}{s}\right) = \frac{\lambda}{2\pi\varepsilon_0}\left(2\pi\,\delta^{(2)}(\vec{s})\right)$$

$\nabla\cdot\vec{E} = \rho/\varepsilon_0$ where $\rho = \lambda\delta^2(\vec{s}) = \lambda\,\delta(x)\delta(y)$

$$Q_{enc}(0 \le z \le L) = \int_{\mathcal{V}_L} \rho\, d\tau = \iint_A \int_0^L \lambda\,\delta(x)\delta(y)\,dz\,dx\,dy$$

(cross-sectional area A)

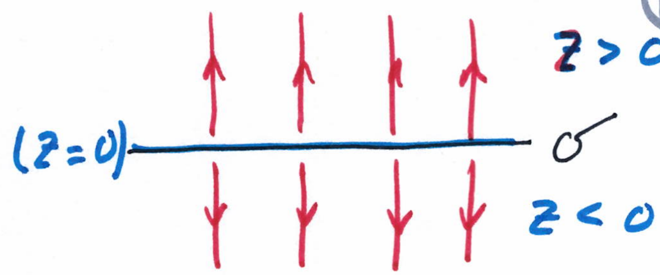
$$= \int_0^L \lambda\, dz$$

$$= \lambda L$$

<u>Density for uniform line charge</u>

$$\rho(x,y,z) = \begin{cases} \infty & \text{if } x = y = 0 \\ 0 & \text{else.} \end{cases}$$

③ $$\vec{E} = \begin{cases} \dfrac{\sigma}{2\varepsilon_0}\hat{z} & z > 0 \\ -\dfrac{\sigma}{2\varepsilon_0}\hat{z} & z < 0 \end{cases}$$

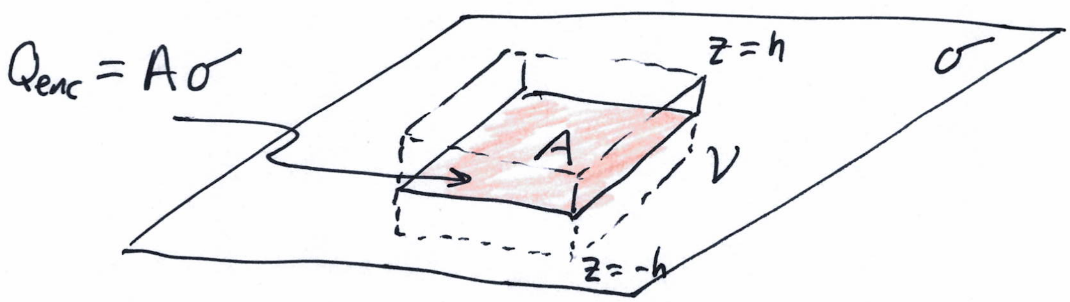


$$\vec{E} = \frac{\sigma}{2\varepsilon_0}\left(2\Theta(z) - 1\right)\hat{z}$$

$$\Theta(z) = \begin{cases} 0 & \text{if } z < 0 \\ 1 & \text{if } z > 0 \end{cases}$$

$$\nabla \cdot \vec{E} = \frac{\sigma}{\varepsilon_0}\frac{d}{dz}\left(\Theta(z)\right) = \frac{\sigma\,\delta(z)}{\varepsilon_0}$$

$$\nabla \cdot \vec{E} = \rho/\varepsilon_0 \quad \text{where} \quad \rho = \sigma\,\delta(z) = \begin{cases} \infty & z = 0 \\ 0 & \text{else.} \end{cases}$$



$$\int_{\mathcal{V}} \rho\, d\tau = \iint_A \int_{-h}^{h} \sigma\,\delta(z)\,dz\,dx\,dy$$

$$= \iint_A \sigma\, dx\,dy$$

$$= \sigma A = Q_{enc} \quad \& \quad \Phi_E = \underbrace{EA}_{\text{top}} + \underbrace{EA}_{\text{bottom}} = 2EA$$

Then $\dfrac{Q_{enc}}{\varepsilon_0} = 2EA = \dfrac{\sigma A}{\varepsilon_0}$ $\therefore$ $\underline{E = \dfrac{\sigma}{2\varepsilon_0}}$.

$\boxed{\text{QUESTION: what about a volume charge?}}$

Problem 1.65 (DIGRESSION ... FOR NOW)

(a.) Check Stokes' Th$^{\underline{m}}$ for $\vec{A} = \dfrac{-y\hat{x}+x\hat{y}}{x^2+y^2}$ using $C_R = \partial D_R$ where D_R is disk in xy-plane with $d\vec{a} = (dxdy)\hat{z}$. "Diagnose problem, and fix it by correcting the $\nabla\times\vec{A}$" (do in Cartesians)

(b.) repeat in sphericals

$$\nabla\times\vec{A} = \left\langle 0,\ 0,\ \frac{\partial}{\partial x}\left[\frac{x}{x^2+y^2}\right] - \frac{\partial}{\partial y}\left[\frac{-y}{x^2+y^2}\right]\right\rangle$$

$$= \left\langle 0,\ 0,\ \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right\rangle$$

$$= \left\langle 0,\ 0,\ \frac{2x^2+2y^2-2x^2-2y^2}{(x^2+y^2)^2}\right\rangle$$

$$= 0.$$ (notice $x=y=0$ blows-up)

Notice C_R has $x = R\cos t$, $y = R\sin t$ for $0 \leq t \leq 2\pi$ hence $dx = -R\sin t\,dt$ and $dy = R\cos t\,dt$ where $x^2+y^2 = R^2$ hence, $d\vec{l} = \langle dx, dy\rangle$?

$$\int_{\partial D_R = C_R} \vec{A}\cdot d\vec{l} = \int_0^{2\pi} \left\langle \frac{-R\sin t}{R^2}, \frac{R\cos t}{R}\right\rangle \cdot \langle -R\sin t, R\cos t\rangle\, dt$$

$$= \int_0^{2\pi} dt$$

$$= 2\pi = \int_{D_R} (\nabla\times\vec{A})\cdot d\vec{a} = \int \left(2\pi\delta^2(\vec{r})\hat{z}\right)\cdot\left(dxdy\,\hat{z}\right)$$

YET, $\displaystyle\int_{D_R} (\nabla\times\vec{A})\cdot d\vec{a} = \int_{D_R} \vec{0}\cdot d\vec{a} = 0.$ (ignoring singularity)

Oh, $\nabla\times\vec{A} = \begin{cases} \infty & \text{if } (x,y) = 0 \\ 0 & \text{if } (x,y) \neq 0 \end{cases} \Rightarrow \boxed{\nabla\times\vec{A} = 2\pi\,\delta^2(\vec{r})\,\hat{z}}$

PROBLEM 1.65 continued

$$\vec{A} = \frac{-y\hat{x} + x\hat{y}}{x^2+y^2} = \frac{1}{\sqrt{x^2+y^2}}\left\langle \frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}} \right\rangle$$

$$= \frac{1}{s}\langle -\sin\phi , \cos\phi \rangle$$

$$= \frac{1}{s}\hat{\varphi}$$

We observe $A_s = 0$ and $A_z = 0$ thus

$$\nabla \times \vec{A} = \left[\frac{1}{s}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\hat{s} + \left[\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right]\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}\left[s A_\phi\right] - \frac{\partial A_s}{\partial \phi}\right]\hat{z}$$

$$\Rightarrow \nabla \times \vec{A} = \frac{1}{s}\frac{\partial}{\partial s}\left[s \frac{1}{s}\hat{\varphi}\right]\hat{z} = 0 \quad \text{for } s \neq 0.$$

Notice

$$\int_{C_R} \vec{A}\cdot d\vec{l} = \int_0^{2\pi} \frac{1}{R}\hat{\varphi}\cdot(R\,d\varphi\,\hat{\varphi}) = \int_0^{2\pi} d\varphi = 2\pi$$

We see $\nabla \times \vec{A} = 2\pi\,\delta^2(\vec{r})\,\hat{z}$ gives

$$\int_{D_R} (\nabla \times \vec{A})\cdot d\vec{a} = \int_{D_R} 2\pi\,\delta^2(\vec{r})\,\hat{z}\cdot(\hat{z}\,\overbrace{dx\,dy}^{d}) = 2\pi$$

$$\int_S (\nabla \times \vec{A})\cdot d\vec{a} = \begin{cases} 0 & \text{if } 0 \notin S \\ 2\pi & \text{if } 0 \in S \end{cases}$$

$$\int_S (2\pi\,\delta^2(\vec{r})\,\hat{z})\cdot d\vec{a} = \begin{cases} 0 & \text{if } 0 \notin S \\ 2\pi & \text{if } 0 \in S. \end{cases}$$