LECTURE 5: INTERIOR, CLOSURE, NEIGHBOURHOODS

Defn/ Let $B \subseteq X$ where X is topological space then the closure of B is denoted $\overline{B}$ where

$$\overline{B} = \bigcap \{ C \mid B \subseteq C \subseteq X,\ C \text{ closed} \}$$

The points in $\overline{B}$ are said to be adherent to B

$\boxed{E1}$ Let $A \subseteq B$ then if $B \subseteq C$ we also have $A \subseteq C \Rightarrow \overline{A} \subseteq \overline{B}$.
Likewise, for family $A_i \subseteq X$ we find $\bigcup_i \overline{A_i} \subseteq \overline{\bigcup_i A_i}$ since
$A_j \subseteq \bigcup_i A_i \Rightarrow \overline{A_j} \subseteq \overline{\bigcup_i A_i} \ \forall j \Rightarrow \bigcup_j \overline{A_j} \subseteq \overline{\bigcup_i A_i}$.

Defn/ $A \subseteq X$ is said to be dense if $\overline{A} = X$.
Equivalently, A meets every non-empty open subset of X.
Generally, if $A \subseteq B$ then A is dense in B when $B \subseteq \overline{A}$

$\boxed{E2}$ 1.) if $\mathcal{T} = \{\emptyset, X\}$ then $\emptyset \neq A \subseteq X$ has $\overline{A} = X$ (indiscrete topology)
2.) if $\mathcal{T} = \mathcal{P}(X)$ then $A \subset X$ has $\overline{A} = A$ (discrete topology)
3.) $\mathbb{Q} \subseteq \mathbb{R}$ with Euclidean top. has $\overline{\mathbb{Q}} = \mathbb{R}$ (rational #s are dense)
4.) in space with $\underbrace{\text{cofinite}}_{C \text{ is closed iff } C = X \text{ or } C \text{ finite}}$ topology any infinite subset is dense.

$\underset{\text{infinite}}{J} \subseteq X$ if $J \subseteq C$ and C is closed then $C \neq$ finite as J infinite
$\Rightarrow C = X \therefore \overline{J} = X$ and J is dense.

Def^{n}/ The <u>interior</u> of $B \subseteq X$ is denoted $B^\circ = \underbrace{\bigcup \{ U \in \mathcal{T} \mid U \subseteq B \}}_{\text{union of all open sets which are subsets of } B.}$

That is, B° is largest open set contained in B.

Points in B° are called <u>interior points</u> of B.

<u>Remark</u>: the definitions for closure and interior used only set theory, no reliance on distance was needed for abstract topological setting.

Th^{m}/ $B \subseteq X$ then B open iff $B = B^\circ$ and B closed iff $B = \overline{B}$.

Moreover, $X - B^\circ = \overline{X - B}$

$\boxed{E3}$ $\overline{(a,b)} = \overline{[a,b)} = \overline{(a,b]} = \overline{[a,b]} = [a,b]$
$(a,b)^\circ = [a,b)^\circ = (a,b]^\circ = [a,b]^\circ = (a,b)$
for $a < b$ in the Euclidean topology on $\mathbb{R}$

Def^{n}/ The <u>boundary</u> of $B \subseteq X$ is denoted ∂B.
Where $\partial B = \overline{B} - B^\circ = \overline{B} \cap (\overline{X - B})$. The "boundary of B contains pts. adherent to both B and $X - B$

$\boxed{E4}$ (again in Euclidean Topology for $\mathbb{R}$)

$\partial [a,b] = \{a,b\}$ $\quad$ $\partial(-\infty,\infty) = \emptyset$
$\partial (a,b) = \{a,b\}$ $\quad$ $\partial(a,\infty) = \{a\}$
$\partial (a,b] = \{a,b\}$ $\quad$ $\partial(-\infty,b) = \{b\}$
$\partial [a,b) = \{a,b\}$

Def^n/ If $x \in X$ is a point in a topological space X then $U \subseteq X$ is a neighbourhood of x (nbhd) if $\exists$ open set V such that $x \in V$ and $V \subseteq U$. In other words, U is a nbhd of x if x is an interior pt. of U

Remark: if $\mathcal{I}(x)$ is the family of nbhds of x then for $A \subseteq X$ we have $A^\circ = \text{interior}(A) = \{x \in A \mid A \in \mathcal{I}(x)\}$. A subset is open iff it is a nbhd of each of its points.

$V = (-0.5, 0.5) \subseteq \underbrace{[-1,1]}_{U}$

ES $[-1,1]$ is nbhd of $0 \in \mathbb{R}$ since ~~$[-1,1] \subseteq (-2,2)$~~ and ~~$0 \in (-2,2)$~~ $0 \in (-0.5, 0.5)$ where ~~$(-2,2)$~~ is open in Euclidean topology on $\mathbb{R}$

Lemma (3.20 Manetti) The family $\mathcal{I}(x)$ of nbhds of a point x is closed under extension(1.) and finite intersection(2.) Meaning:

(1.) if $U \in \mathcal{I}(x)$ and $U \subseteq V$ then $V \in \mathcal{I}(x)$

(2.) if $U, V \in \mathcal{I}(x)$ then $U \cap V \in \mathcal{I}(x)$

Proof (1.) Let $U \in \mathcal{I}(x)$ and suppose $U \subseteq V$. By def^n $\exists V_0$ open in X with $x \in V_0$ and $V_0 \subseteq U$. But $V_0 \subseteq U \subseteq V \therefore V_0 \subseteq V$ and we find $V \in \mathcal{I}(x)$.

(2.) if $U, V \in \mathcal{I}(x)$ then $\exists$ open $U_0, V_0 \subseteq X$ for which $x \in U_0$, $x \in V_0$ and $U_0 \subseteq U$ and $V_0 \subseteq V$. Observe $x \in U_0 \cap V_0$ and $U_0 \cap V_0$ is intersection of open sets which is open by A3. Moreover, $U_0 \cap V_0 \subseteq U \cap V$ thus $U \cap V \in \mathcal{I}(x)$. //

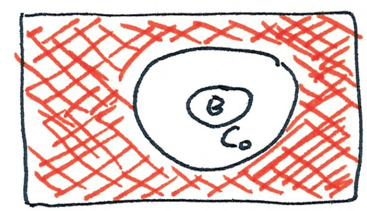
Lemma: Let B be a subset of top. space X. A point $x \in X$ belongs to $\bar{B} = \text{closure}(B)$ iff $U \cap B \neq \emptyset$ for every nbhd $U \in \mathcal{I}(x)$

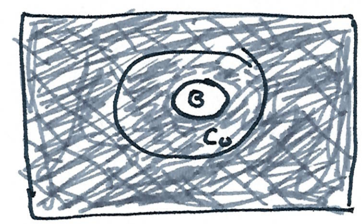
Proof: Recall $x \notin \bar{B}$ iff x is an interior point of $X - B$

$x \in \bar{B} = \bigcap_{\substack{C \text{ closed} \\ B \subseteq C}} C \Rightarrow x \in C$ $\forall$ closed sets C with $B \subseteq C$

$x \notin \bar{B} \Rightarrow \exists C_0$ closed set with $B \subseteq C_0$ and $x \notin C_0$

Hence $x \in \underline{X - C_0} \subseteq \underline{X - B}$





Then $x \in (X - B)^\circ$ iff $\exists\, U \in \mathcal{I}(x)$ such that $U \subseteq X - B$. The Lemma follows.//

$\boxed{\text{Def}^{n}/ \text{ Let } x \in X. \text{ A subfamily } \mathcal{J} \subseteq \mathcal{I}(x) \text{ is called a \underline{local basis of nbhds} at } x \text{ if for any } U \in \mathcal{I}(x) \text{ there exists } A \in \mathcal{J} \text{ such that } A \subseteq U.}$

$\boxed{\text{EG}}$ 1.) if we fix $U_0 \in \mathcal{I}(x)$ for a given pt. x in a topological space X then all $U \in \mathcal{I}(x)$ for which $U \subseteq U_0$ form a local basis at x.

2.) if $\mathcal{B}$ is a <u>basis</u> for the given topological space X then open sets in $\mathcal{B}$ which contain x form local basis at x.
(see Th^{m} 3.7 for why (2.) true)

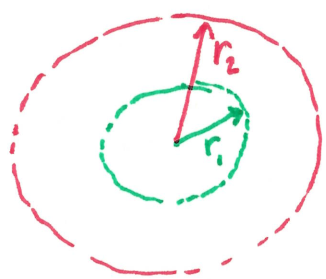
$\boxed{E7}$ Let $X = \mathbb{R}^2$ and $\mathcal{J} = \{\emptyset, \mathbb{R}^2, D_r \mid r > 0\}$ (Ex. 3.10 on pg. 45 of Manetti)

where $D_r = \{(x,y) \mid x^2+y^2 < r^2\}$. Is $\mathcal{J}$ a topology for $\mathbb{R}^2$?

Clearly, $\emptyset, \mathbb{R}^2 \in \mathcal{J}$.

Suppose $D_{r_1}, D_{r_2} \in \mathcal{J}$ then let $R = \max\{r_1, r_2\}$ and $r = \min\{r_1, r_2\}$

then $D_{r_1} \cap D_{r_2} = D_r$ whereas $D_{r_1} \cup D_{r_2} = D_R$



for $r_1 \le r_2$ we have $D_{r_1} \subseteq D_{r_2}$ and thus $D_{r_1} \cup D_{r_2} = D_{r_2}$ whereas $D_{r_1} \cap D_{r_2} = D_{r_1}$

Consider an arbitrary union of $D_r \in \mathcal{J}$ ($\mathbb{R}^2, \emptyset$ left to reader)

If $\bigcup_{r \in I} D_r$ and I is not bounded above then $\bigcup_{r \in I} D_r = \mathbb{R}^2$

If I is bounded above then $\bigcup_{r\in I} D_r = D_{\mathrm{lub}(I)} \in \mathcal{J}$. Thus $\mathcal{J}$ forms topology for $\mathbb{R}^2$. (e.g. $\bigcup_{r \in (3,10)} D_r = D_{10}$ etc.)

What is the closure of $S = \{(x,y) \mid xy = 1\}$?

$$\bar{S} = \bigcap_{\substack{\text{closed sets } C \\ S \subseteq C}} C = \bigcap_{r \le \sqrt{2}} (\mathbb{R}^2 - D_r) = \mathbb{R}^2 - D_{\sqrt{2}}$$