LECTURE 6 7: POTENTIAL AND CONDUCTORS

Given source charge with density ρ the electric field at $\vec{r}$ is given by the following where $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$

$$\boxed{\vec{E}(\vec{r}) = \int \frac{\rho(\vec{r}\,')\,d\tau'}{\mathscr{r}^2}\hat{\mathscr{r}}}$$

where the integration is over all of space.

$$\boxed{\underline{\text{Lemma}}: \quad \nabla \times \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 0}$$

see over

Therefore, as $\nabla = \hat{x}\,\partial_x + \hat{y}\,\partial_y + \hat{z}\,\partial_z$ we find

$$\boxed{\nabla \times \vec{E} = \int \left(\rho(\vec{r}\,')\,d\tau\right) \nabla \times \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 0.}$$

Thus, wherever $\vec{E}$ is defined, $\nabla \times \vec{E} = 0$. It follows $\int_C \vec{E} \cdot d\vec{l}$ is independent of path,

Remark: $\nabla \times \vec{F} = 0$ on a simply connected domain gives path-independence and hence the existence of f s.t. $\vec{F} = \nabla f$. We do need dom$(\vec{E})$ simply connected. Can you think of a distribution which would spoil the simple connectedness of domain for $\vec{E}$?

$\boxed{\underline{\text{Lemma}}: \ \nabla \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = 0}$ Proof

$$\vec{F} = \frac{\vec{\mathscr{r}}}{\mathscr{r}^3} \qquad \vec{\mathscr{r}} = \langle x - x', y - y', z - z' \rangle$$

$$F_j = \frac{x_j - x_j'}{\left((x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2\right)^{3/2}} = \frac{x_j - x_j'}{\mathscr{r}^3}$$

$$\frac{\partial \mathscr{r}}{\partial x_i} = \frac{\partial}{\partial x_i}\left[\sqrt{(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2}\right] = \frac{x_i - x_i'}{\mathscr{r}}$$

$$\frac{\partial F_j}{\partial x_i} = \frac{\partial}{\partial x_i}\left[\frac{x_j - x_j'}{\mathscr{r}^3}\right] = \frac{\delta_{ij}}{\mathscr{r}^3} + (x_j - x_j')\left(\frac{-3}{\mathscr{r}^4}\right)\frac{\partial \mathscr{r}}{\partial x_i}$$

$$\frac{\partial F_j}{\partial x_i} = \frac{\delta_{ij}}{\mathscr{r}^3} - \frac{3(x_j - x_j')(x_i - x_i')}{\mathscr{r}^5} =$$

$$= \frac{\delta_{ji}}{\mathscr{r}^3} - \frac{3(x_i - x_i')(x_j - x_j')}{\mathscr{r}^5}$$

$$= \frac{\partial F_i}{\partial x_j}$$

Thus, $\partial_i F_j = \partial_j F_i$ and we find

$$\nabla \times \vec{F} = \sum_{i,j,k} \epsilon_{ijk}(\partial_i F_j)\hat{x}_k$$

$$= \sum_{j,i,k} \epsilon_{jik}(\partial_j F_i)\hat{x}_k$$

$$= -\sum_{i,j,k} \epsilon_{ijk}(\partial_i F_j)\hat{x}_k$$

$$= -\nabla \times \vec{F} \quad \therefore \quad \underline{\nabla \times \vec{F} = 0}.$$

Since $\nabla \times \vec{E} = 0$ we can be sure there exists a potential function V on simply connected regions of space such that $\vec{E} = -\nabla V$.

Def Given an electric field $\vec{E}$ we say V is a <u>potential</u> for $\vec{E}$ if $\vec{E} = -\nabla V$

Let's examine a few simple examples,

$\boxed{E1}$ $\vec{E} = E_o \hat{z}$ has $V(x,y,z) = -E_o z$. since

$\nabla(-E_o z) = -E_o \nabla z = -E_o \hat{z}$ (we assume E_o constant)

thus $\vec{E} = -\nabla V$.

$\boxed{E2}$ $\vec{E} = \dfrac{\lambda}{2\pi\varepsilon_o}\dfrac{\hat{s}}{s}$ was the electric field for an infinite uniform line charge λ along z-axis.

$$\nabla V = \hat{s}\frac{\partial V}{\partial s} + \frac{1}{s}\frac{\partial V}{\partial \phi}\hat{\phi} + \frac{\partial V}{\partial z}\hat{z} = \left(\frac{-\lambda}{2\pi\varepsilon_o}\frac{1}{s}\right)\hat{s}$$

Thus we must solve, we see $V = V(s)$,

$$\frac{\partial V}{\partial s} = \frac{-\lambda}{2\pi\varepsilon_o}\frac{1}{s}$$

$$V(s) = \frac{-\lambda}{2\pi\varepsilon_o}\ln(s)$$

<u>Remark</u>: the potential is zero for $\boxed{E1}$ on the whole xy-plane where $z = 0$. In $\boxed{E2}$ the potential is zero on the cylinder $s = 1$. Both $\boxed{E1}$ & $\boxed{E2}$ correspond to charge dist. extending only far out.

E3 Point charge Q at $\vec{r}\,'$ has Coulomb field

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0}\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

where $\vec{\mathscr{r}} = \langle x-x', y-y', z-z' \rangle$ then $\vec{E} = -\nabla V$ gives

$$\frac{-Q}{4\pi\varepsilon_0}\frac{1}{\mathscr{r}^3}\langle x-x', y-y', z-z' \rangle = \left\langle \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} \right\rangle$$

We wish to solve:

$$* \begin{cases} \dfrac{\partial V}{\partial x} = \dfrac{-Q(x-x')}{4\pi\varepsilon_0\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{3/2}} \\[2ex] \dfrac{\partial V}{\partial y} = \dfrac{-Q(y-y')}{4\pi\varepsilon_0\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{3/2}} \\[2ex] \dfrac{\partial V}{\partial z} = \dfrac{-Q(z-z')}{4\pi\varepsilon_0\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{3/2}} \end{cases}$$

Notice $\mathscr{r}^2 = (x-x')^2+(y-y')^2+(z-z')^2$ gives us

$$2\mathscr{r}\frac{\partial \mathscr{r}}{\partial x_i} = 2(x_i - x_i') \hookrightarrow \underline{\frac{\partial \mathscr{r}}{\partial x_i} = \frac{x_i - x_i'}{\mathscr{r}}}$$

Ok, let's look at $*$ with x_i-notation,

$$\frac{\partial V}{\partial x_i} = \frac{-Q(x_i - x_i')}{4\pi\varepsilon_0 \mathscr{r}^3} = \frac{-Q}{4\pi\varepsilon_0\mathscr{r}^2}\frac{\partial \mathscr{r}}{\partial x_i}$$

$$\Rightarrow \boxed{V = \frac{Q}{4\pi\varepsilon_0 \mathscr{r}}}$$

<u>Remark</u>: this is much easier if we consider $\vec{r}\,' = 0$

E4 Find potential for Q at origin, this time we are able to use <u>spherical coordinates</u> (5)

$$\vec{E} = \frac{Q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2} = -\nabla V = -\frac{\partial V}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} - \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi}$$

We find $\frac{\partial V}{\partial \theta} = \frac{\partial V}{\partial \phi} = 0$ hence $V = V(r)$ and,

$$\frac{\partial V}{\partial r} = \frac{dV}{dr} = \frac{-Q}{4\pi\epsilon_0 r^2} \Rightarrow \underline{V(r) = \frac{Q}{4\pi\epsilon_0 r}}.$$

Often calculation of potential from integration of $\vec{E} = -\nabla V$ is a wise method to calculate V, especially if $\vec{E}$ can be calculated via symmetry-based Gauss Law arguments. We should review some relevant <u>vector calculus</u>

> **Th^m/** Given simply connected subset $S \subseteq \mathbb{R}^3$ the following are equivalent,
>
> (1.) $\vec{F} = \nabla f$ on S ($\vec{F}$ is <u>conservative vector field</u>)
>
> (2.) $\int_{C_1} \vec{F} \cdot d\vec{l} = \int_{C_2} \vec{F} \cdot d\vec{l}$ for coterminal paths C_1, C_2 inside S
>
> (3.) $\oint_{loop} \vec{F} \cdot d\vec{l} = 0$ for loops in S
>
> (4.) $\nabla \times \vec{F} = 0$
>
> Moreover, if $\vec{r}_o$ is some point in S then
>
> $$f(\vec{r}) = \int_{\vec{r}_o}^{\vec{r}} \vec{F} \cdot d\vec{l}$$

Since $\nabla \times \vec{E} = 0$ we have $\oint_{loop} \vec{E} \cdot d\vec{l} = 0$ and provided we adjust the formula above with appropriate minus we also can calculate V by calculating line-integral

Provided $\rho(r') = 0$ for r' sufficiently large, that is supposing the charge distribution is local and does not extend to $r' \longrightarrow \infty$, we may set the potential $V(\infty) = 0$ and calculate the potential via the following:

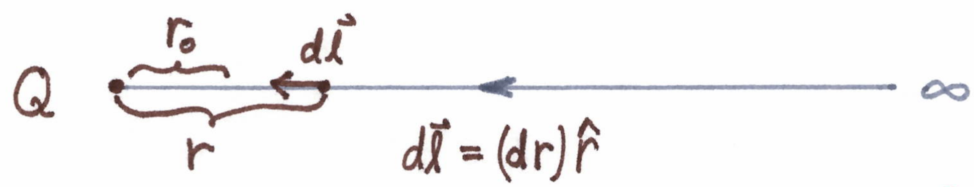
$$\boxed{V(\vec{r}) = -\int_{\infty}^{\vec{r}} \vec{E} \cdot d\vec{l}}$$

Notice we may also take some point $\mathcal{O}$ as the <u>zero</u> for potential

$$\boxed{V(r) = -\int_{\mathcal{O}}^{\vec{r}} \vec{E} \cdot d\vec{l}}$$

But, it's nice to use ∞ when it's possible.

E5 Consider point charge at origin,



$$d\vec{l} = (dr)\hat{r}$$

$$\vec{E} = \frac{Q}{4\pi\varepsilon_o}\frac{\hat{r}}{r^2}$$

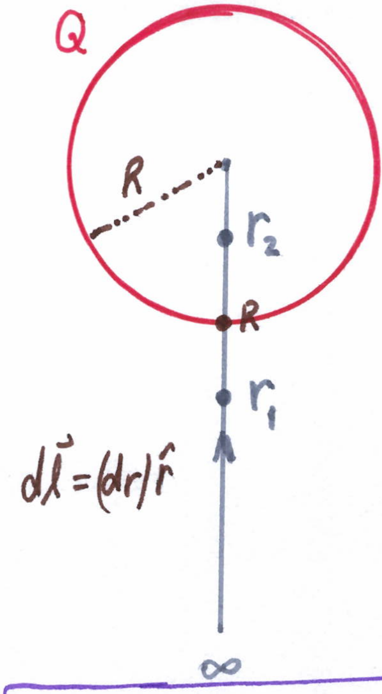
$$\vec{E} \cdot d\vec{l} = \frac{Q}{4\pi\varepsilon_o}\frac{dr}{r^2}$$

<u>Remark</u>: notice on path from ∞ to $\vec{r}_o$ we <u>decrease</u> r hence $dr < 0$, so $d\vec{l} = (dr)\hat{r}$ points <u>towards</u> the origin.

$$\begin{aligned}
V(r_o) &= -\int_{\infty}^{\vec{r}_o} \vec{E} \cdot d\vec{l} \\
&= -\int_{\infty}^{r_o} \frac{Q}{4\pi\varepsilon_o}\frac{dr}{r^2} \\
&= \frac{Q}{4\pi\varepsilon_o}\frac{1}{r}\Big|_{\infty}^{r_o} \\
&= \frac{Q}{4\pi\varepsilon_o r_o}
\end{aligned}$$

$$\therefore \quad \boxed{V(r) = \frac{Q}{4\pi\varepsilon_o r}}$$

E6 Find voltage for spherical shell charge Q spread uniformly on sphere of radius R



$$\vec{E} = \begin{cases} 0 & \text{for } r < R \\ \dfrac{Q}{4\pi\varepsilon_0}\dfrac{\hat{r}}{r^2} & \text{for } r > R \end{cases}$$

For $r_1 > R$,

$$V(r_1) = -\int_{\infty}^{r_1} \vec{E}\cdot d\vec{l} = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r_1}$$

For $r_2 < R$,

$$V(r_2) = -\int_{\infty}^{r_2} \vec{E}\cdot d\vec{l} = -\int_{\infty}^{R} \vec{E}\cdot d\vec{l} - \int_{R}^{r_2} \vec{E}\cdot d\vec{l}$$

$$\therefore \underbrace{V(r_2) = V(R) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{R}}$$

to be fussy, we have to take limit to approach $r = R$, but continuity of V requires we set $V(R) = \dfrac{Q}{4\pi\varepsilon_0}\dfrac{1}{R}$

$$\boxed{V(r) = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0}\dfrac{1}{R} & : r \le R \\ \dfrac{Q}{4\pi\varepsilon_0}\dfrac{1}{r} & : r > R \end{cases}}$$

E7 Consider uniformly charged sphere of radius R we derived by Gauss' Law

$$\vec{E} = \begin{cases} \dfrac{Qr\hat{r}}{4\pi\varepsilon_0 R^3} & : 0 \le r < R \\ \dfrac{Q\hat{r}}{4\pi\varepsilon_0 r^2} & : r \ge R \end{cases}$$

Thus, for $r_0 < R$, once more $d\vec{l} = \hat{r}\,dr$ and,

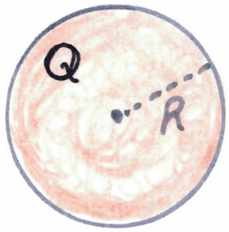
$$V(r_0) = -\int_{\infty}^{r_0} \vec{E}\cdot d\vec{l} = -\int_{\infty}^{R} \frac{Q\hat{r}}{4\pi\varepsilon_0 r^2}\cdot\hat{r}\,dr - \int_{R}^{r_0} \frac{Qr\hat{r}}{4\pi\varepsilon_0 R^3}\cdot\hat{r}\,dr$$

$$\therefore V(r_0) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r}\Big|_{\infty}^{R} - \left(\frac{1}{2}r^2\Big|_{R}^{r_0}\right)\frac{Q}{4\pi\varepsilon_0 R^3}$$

$$V(r_0) = \frac{Q}{4\pi\varepsilon_0 R} + \frac{Q}{4\pi\varepsilon_0 R^3}\left(\frac{R^2}{2} - \frac{r_0^2}{2}\right)$$

$$V(r) = \frac{3Q}{8\pi\varepsilon_0 R} - \frac{Qr^2}{8\pi\varepsilon_0 R^3} = \frac{Q}{8\pi\varepsilon_0}\left[\frac{3}{R} - \frac{r^2}{R^3}\right]$$

<u>E7 continued</u>: the potential depends on r alone and,



$$V(r) = \begin{cases} \dfrac{Q}{8\pi\varepsilon_0}\left[\dfrac{3}{R} - \dfrac{r^2}{R^3}\right] & : r \leq R \\[2ex] \dfrac{Q}{4\pi\varepsilon_0}\dfrac{1}{r} & : r \geq R \end{cases}$$

outside the sphere the potential is same as if the entire charge was concentrated at $r = 0$. In other words, sufficiently far away a localized charge has potential resembling that of the Coulomb field. I often call $\dfrac{Q}{4\pi\varepsilon_0}\dfrac{1}{\mathscr{r}}$ the Coulomb potential

Thm/ Superposition of Potentials
If ρ_1 gives potential V_1 and ρ_2 gives potential V_2 then $V_1 + V_2$ serves as potential for $\rho = \rho_1 + \rho_2$

<u>Proof</u>: V_1 gives $\vec{E}_1 = -\nabla V_1$ for which $\nabla \cdot \vec{E}_1 = \rho_1/\varepsilon_0$ likewise $\vec{E}_2 = -\nabla V_2$ with $\nabla \cdot \vec{E}_2 = \rho_2/\varepsilon_0$. Then,

$$-\nabla(V_1 + V_2) = -\nabla V_1 - \nabla V_2 = \vec{E}_1 + \vec{E}_2$$

and

$$\nabla \cdot (\vec{E}_1 + \vec{E}_2) = \nabla \cdot \vec{E}_1 + \nabla \cdot \vec{E}_2 = \frac{\rho_1}{\varepsilon_0} + \frac{\rho_2}{\varepsilon_0}$$

Hence $V = V_1 + V_2$ is potential for charge distribution $\rho = \rho_1 + \rho_2$.

<u>Remark</u>: if $V_1(\infty) = 0$ and $V_2(\infty) = 0$ then $V_1 + V_2 = V$ also has $V(\infty) = 0$.

E8

Given shell of charge at $r = R$ of Q_2 and uniform ball of charge on $0 \le r \le R$ of Q_1 we can write down potential from $\boxed{E6}$ and $\boxed{E7}$ and superposition,

$$V(r) = V_{E6}(r) + V_{E7}(r) = \begin{cases} \dfrac{Q_1}{8\pi\epsilon_0}\left[\dfrac{3}{R} - \dfrac{r^2}{R^3}\right] + \dfrac{Q_2}{4\pi\epsilon_0 R} & : \ 0 \le r \le R \\ \dfrac{Q_1 + Q_2}{4\pi\epsilon_0}\dfrac{1}{r} & : \ r \ge R \end{cases}$$

We have shown $V(\vec{r}) = \dfrac{Q}{4\pi\epsilon_0}\dfrac{1}{\mathscr{r}}$ for Q at $\vec{r}\,'$ thus if we consider an infinitesimal charge dQ at $\vec{r}\,'$ then naturally $dV = \dfrac{dQ}{4\pi\epsilon_0}\dfrac{1}{\mathscr{r}}$ where $\mathscr{r} = \|\vec{r} - \vec{r}\,'\|$ thus,

$$\boxed{V(\vec{r}) = \int \frac{dQ}{4\pi\epsilon_0 \mathscr{r}} \quad \text{for local distribution of } dQ}$$

As we saw in the formulation of $\vec{E}$ from integration of charge distribution we have cases to consider,

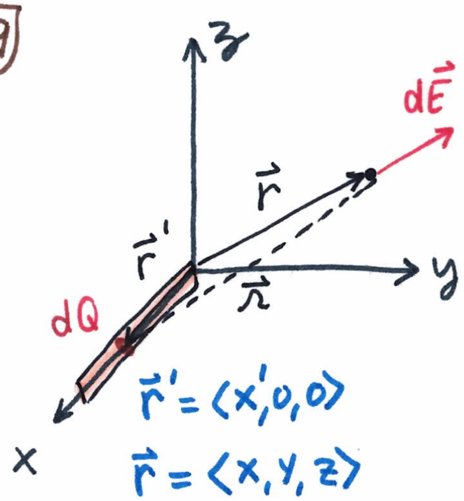
LINEAR CHARGE DISTRIBUTION λ : $V(\vec{r}) = \displaystyle\int \frac{\lambda\, d\ell'}{4\pi\epsilon_0 \mathscr{r}}$

SURFACE CHARGE DISTRIBUTION σ : $V(\vec{r}) = \displaystyle\int \frac{\sigma\, da'}{4\pi\epsilon_0 \mathscr{r}}$

VOLUME CHARGE DISTRIBUTION ρ : $V(\vec{r}) = \displaystyle\int \frac{\rho\, d\tau'}{4\pi\epsilon_0 \mathscr{r}}$

$$\mathscr{r} = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

E9



$\vec{r}' = \langle x', 0, 0 \rangle$

$\vec{r} = \langle x, y, z \rangle$

Consider Q uniformly distributed on x-axis from $(0,0,0)$ to $(L,0,0)$

$$dV = \frac{dQ}{4\pi\varepsilon_0} \frac{1}{\sqrt{(x-x')^2 + y^2 + z^2}}$$

$$dQ = \lambda\, dx' = \frac{Q}{L} dx'$$

$$V(x,y,z) = \frac{Q}{4\pi\varepsilon_0 L} \int_0^L \frac{dx'}{\sqrt{(x'-x)^2 + y^2 + z^2}}$$

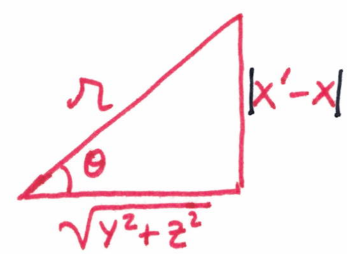
$x' = x + \tan\theta \sqrt{y^2+z^2}$

$dx' = \sec^2\theta \sqrt{y^2+z^2}\, d\theta$

$$= \frac{Q}{4\pi\varepsilon_0 L} \int_{x'=0}^{x'=L} \frac{\sec^2\theta \sqrt{y^2+z^2}\, d\theta}{\sqrt{\left(\tan\theta \sqrt{y^2+z^2}\right)^2 + y^2 + z^2}}$$

$\tan\theta = \dfrac{|x'-x|}{\sqrt{y^2+z^2}}$

$$= \frac{\lambda}{4\pi\varepsilon_0} \int_{x'=0}^{x'=L} \frac{\sqrt{y^2+z^2}\, \sec^2\theta\, d\theta}{\sqrt{y^2+z^2}\, \sec\theta}$$



$$= \frac{\lambda}{4\pi\varepsilon_0} \ln|\sec\theta + \tan\theta| \Big|_{x'=0}^{x'=L}$$

$\underline{x'=L:}$

$\mathscr{r} = \sqrt{(L-x)^2 + y^2 + z^2}$

$$= \frac{\lambda}{4\pi\varepsilon_0} \ln\left|\frac{\sec\theta_L + \tan\theta_L}{\sec\theta_0 + \tan\theta_0}\right|$$

$\sec\theta_L = \dfrac{\sqrt{(x-L)^2 + y^2 + z^2}}{\sqrt{y^2+z^2}}$

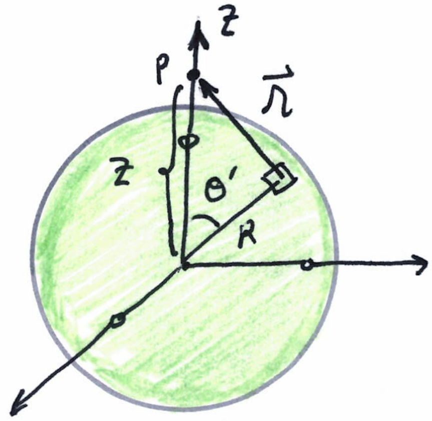
$$= \frac{\lambda}{4\pi\varepsilon_0} \ln\left|\frac{\left(\dfrac{|x-L| + \sqrt{(x-L)^2 + y^2 + z^2}}{\sqrt{y^2+z^2}}\right)}{\dfrac{\sqrt{y^2+z^2+L^2}}{\sqrt{y^2+z^2}} + \dfrac{|x|}{\sqrt{y^2+z^2}}}\right|$$

$\tan\theta_L = \dfrac{|x-L|}{\sqrt{y^2+z^2}}$

$$= \underbrace{\boxed{\frac{\lambda}{4\pi\varepsilon_0} \ln\left|\frac{|x-L| + \sqrt{(x-L)^2 + y^2 + z^2}}{|x| + \sqrt{L^2 + y^2 + z^2}}\right|}}_{V(x,y,z)}$$

$\boxed{E10}$ Potential of uniformly charged spherical shell of radius R with density σ $(Q = 4\pi R^2 \sigma)$
This is Example 2.8 in Griffiths.

$$\frac{\sigma R}{\varepsilon_0} = \frac{QR}{4\pi\varepsilon_0 R^2} = \frac{Q}{4\pi\varepsilon_0}$$



$$\mathscr{r}^2 = R^2 + z^2 - 2Rz\cos\theta'$$

$$dQ = \sigma\, da' = \sigma R^2 \sin\theta'\, d\theta'\, d\phi'$$

using θ', ϕ' to range over the shell of charge at $r = R$.

$$V(P) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma\, da'}{\mathscr{r}}$$

$$= \frac{\sigma}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^{\pi} \frac{R^2 \sin\theta'\, d\theta'\, d\phi'}{\sqrt{R^2+z^2-2Rz\cos\theta'}}$$

$$= \frac{2\pi R^2 \sigma}{4\pi\varepsilon_0}\int_0^{\pi} \frac{\sin\theta'\, d\theta'}{\sqrt{R^2+z^2-2Rz\cos\theta'}}$$

$u = R^2+z^2-2Rz\cos\theta'$
$du = +2Rz\sin\theta'\,d\theta'$
$\sin\theta'\,d\theta' = \dfrac{du}{2Rz}$

$$= \frac{R^2\sigma}{2\varepsilon_0}\,\frac{1}{Rz}\sqrt{R^2+z^2-2Rz\cos\theta'}\,\Big|_0^{\pi}$$

$$= \frac{\sigma R}{2\varepsilon_0 z}\left[\sqrt{R^2+z^2+2Rz} - \sqrt{R^2+z^2-2Rz}\right]$$

$$= \frac{\sigma R}{2\varepsilon_0 z}\left[\sqrt{(z+R)^2} - \sqrt{(z-R)^2}\right]$$

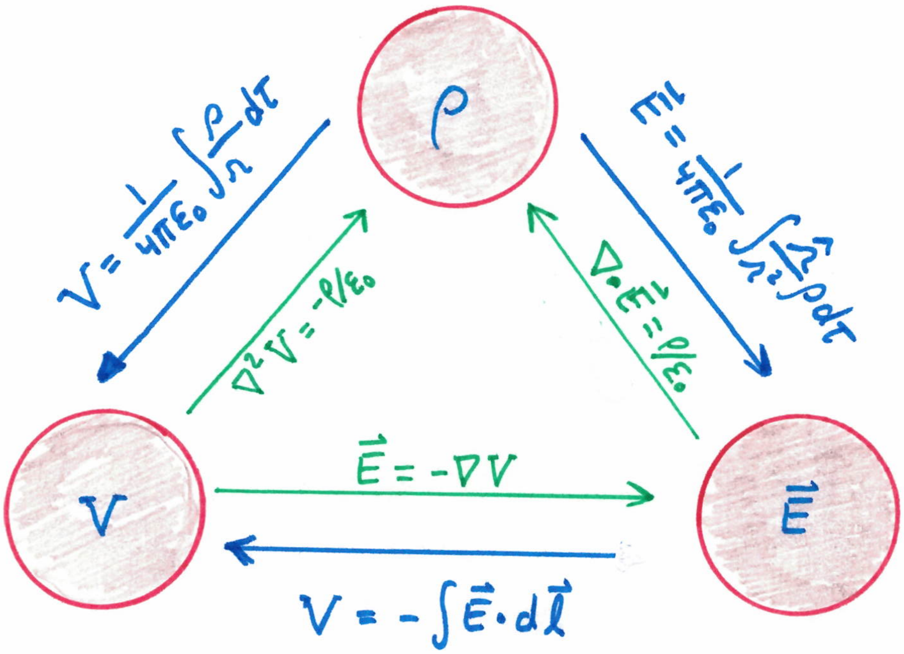
$$= \frac{\sigma R}{2\varepsilon_0 z}\left[\,|z+R| - |z-R|\,\right]$$

$$= \frac{\sigma R}{2\varepsilon_0 z}\begin{cases} z+R-(R-z) & : \ 0 \le z \le R \\ z+R-(z-R) & : \ z \ge R \end{cases}$$

$$= \frac{\sigma R}{2\varepsilon_0 z}\begin{cases} 2z & : \ 0 \le z \le R \\ 2R & : \ z \ge R \end{cases} \qquad \therefore \ \boxed{V(z) = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0 R} & : 0 \le z \le R \\ \dfrac{Q}{4\pi\varepsilon_0 z} & : z \ge R \end{cases}}$$

Properties of $\vec{E}$ and V and $\rho = \frac{dQ}{d\tau}$

Griffiths offers a diagram to illustrate the inter dependencies of $\vec{E}$, V, and ρ.



Remark: I've omitted $\nabla\times\vec{E} = 0$ from the diagram, it seems to me it belongs in at least two places since,

① $\vec{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\gamma}}{\gamma^2}\rho\, d\tau \Rightarrow \nabla\times\vec{E} = 0$ (integrate the lemma earlier)

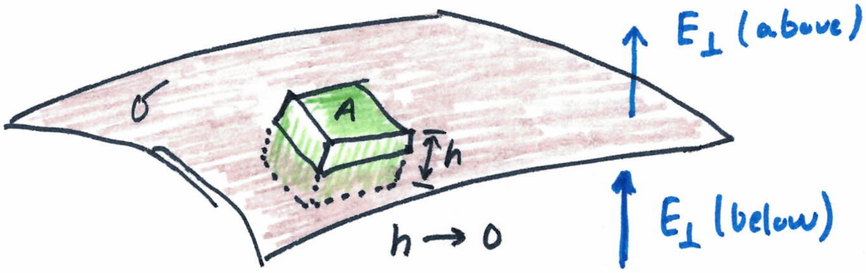
② $\vec{E} = -\nabla V \Rightarrow \nabla\times\vec{E} = 0$ as $\nabla\times\nabla V = 0$.

Key Features of $\vec{E}$ in electrostatics

$$\underbrace{\int \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0} \quad \text{and} \quad \oint \vec{E}\cdot d\vec{l} = 0}_{\text{we examine what these imply about the values of } \vec{E} \text{ and } V \text{ as we go above/below a given surface.}}$$

Boundary Conditions for $\vec{E}$ and V

Consider surface with surface charge density σ

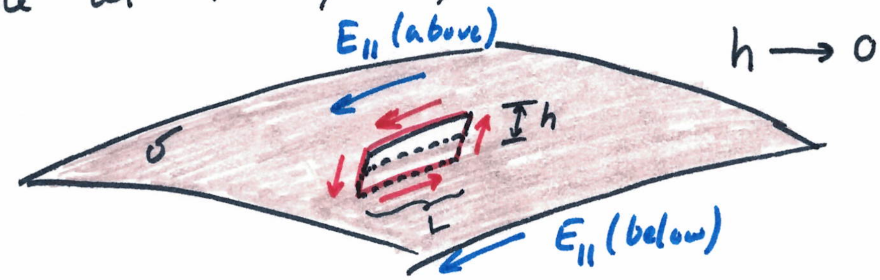


we imagine a thin-volume with cross-sectional area A then Gauss' Law yields:

$$\Phi_E = AE_\perp(\text{above}) - AE_\perp(\text{below}) = \frac{\sigma A}{\varepsilon_0} = \frac{Q_{enc}}{\varepsilon_0}$$

$$\boxed{E_\perp(\text{above}) - E_\perp(\text{below}) = \frac{\sigma}{\varepsilon_0}}$$

Now imagine a loop around the same point such that the top and bottom legs are parallel to the surface at the point,



$$\int_C \vec{E}\cdot d\vec{l} = 0 = E_\parallel(\text{above})L - E_\parallel(\text{below})L$$

$$\therefore \quad \underbrace{\boxed{E_\parallel(\text{above}) = E_\parallel(\text{below})}}$$

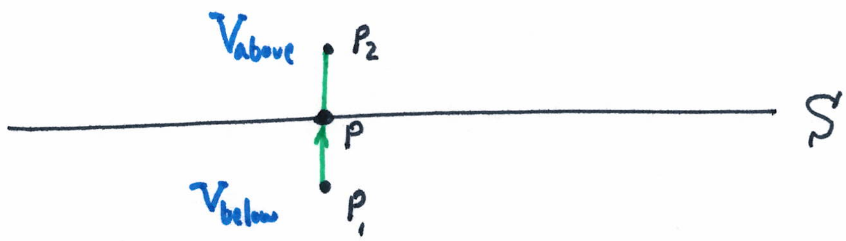
Holds for any direction tangent to surface at given point.

$$\boxed{\text{Th}^{m}/ \quad \vec{E}_{\text{above}} - \vec{E}_{\text{below}} = \frac{\sigma}{\varepsilon_0}\hat{n}}$$

} here $\hat{n}$ defines the "above" direction.

Boundary Conditions for V

Consider a boundary S and imagine a path from P_1 to P_2 where P_1 is just below S and P_2 is just above S. I'll use 2D-picture



$$V_{above} - V_{below} = -\int_{P_1}^{P_2} \vec{E} \cdot d\vec{l} \longrightarrow 0 \quad \text{as } P_1, P_2 \longrightarrow P.$$

$$\therefore \quad \boxed{V_{above} = V_{below}}$$

Remark: S above is totally arbitrary, we can justly expect potential must be everywhere continuous. (excluding wild ideas like ∞ charge at a point)

We argued $\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n}$ but we know $\vec{E} = -\nabla V$ hence we find,

$$\boxed{\nabla V_{above} - \nabla V_{below} = \frac{-\sigma}{\varepsilon_0}\hat{n}}$$

$$\underline{\frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = \frac{-\sigma}{\varepsilon_0}}$$

Def^{n}/ $\frac{\partial f}{\partial u} = (\nabla f) \cdot \hat{u}$ hence $\underbrace{\frac{\partial f}{\partial n} = (\nabla f) \cdot \hat{n}}_{\text{normal derivative.}}$

WORK AND ENERGY IN ELECTROSTATICS (§2.4 Griffiths)

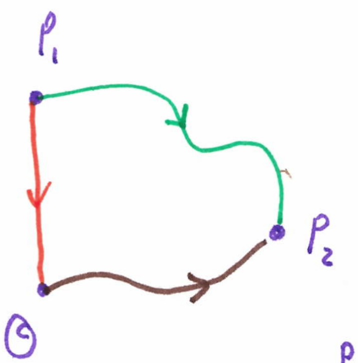
Given a configuration of charge which produces electric field $\vec{E}$, if we move Q from P_1 to P_2 using minimal effort we have $\vec{F}_{push} + \vec{F}_{electric} = m\vec{a} = 0$ (just barely moving $\Rightarrow$ $\vec{F}_{push} = -\vec{F}_{electric} = -Q\vec{E}$

$$W = \int_{P_1}^{P_2} \vec{F}_{push} \cdot d\vec{l} = -Q\int_{P_1}^{P_2} \vec{E}\cdot d\vec{l} = Q\left(-\int_{P_1}^{P_2} \vec{E}\cdot d\vec{l}\right)$$

$$\Rightarrow \quad W = Q\left(-\int_{\mathcal{O}}^{P_2} \vec{E}\cdot d\vec{l} + \int_{\mathcal{O}}^{P_1} \vec{E}\cdot d\vec{l}\right)$$ (see below for details)

$$\Rightarrow \quad \underbrace{\boxed{W = Q\left(V(P_2) - V(P_1)\right)}}$$

If we do not increase KE of Q then work to move Q from P_1 to P_2 against $Q\vec{E}$ is given by $W = Q\Delta V$ where $\Delta V = V(P_2) - V(P_1)$



$$\therefore \quad \boxed{V(P_2) - V(P_1) = \frac{W}{Q}}$$

path-indep. of $\vec{E}$

Remark: $\underbrace{\int_{P_1}^{P_2} \vec{E}\cdot d\vec{l}} = \underbrace{\int_{P_1}^{\mathcal{O}} \vec{E}\cdot d\vec{l}} + \underbrace{\int_{\mathcal{O}}^{P_2} \vec{E}\cdot d\vec{l}} = \int_{\mathcal{O}}^{P_2} \vec{E}\cdot d\vec{l} - \int_{\mathcal{O}}^{P_1} \vec{E}\cdot d\vec{l}$

Therefore, $-\int_{P_1}^{P_2} \vec{E}\cdot d\vec{l} = -\int_{\mathcal{O}}^{P_2} \vec{E}\cdot d\vec{l} - \left(-\int_{\mathcal{O}}^{P_1} \vec{E}\cdot d\vec{l}\right) = V(P_2) - V(P_1)$.

Thm/ work to bring Q from far away ~~is given~~ to $\vec{r}$ by $W = Q[V(\vec{r}) - V(\infty)]$ hence if $V(\infty) = 0$ we have $\underbrace{W = QV(\vec{r})}$

potential is PE per unit-charge.

ENERGY TO CREATE CHARGE DISTRIBUTION

Imagine bringing $q_1, q_2, \dots, q_n$ to positions $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n$ then we can argue the work required is given by

WHY? SEE BELOW.

$$\boxed{W = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{j>i}^{n}\frac{q_i q_j}{r_{ij}} = \frac{1}{8\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{j\neq i}\frac{q_i q_j}{r_{ij}}}$$

where $r_{ij} = \|\vec{r}_i - \vec{r}_j\|$ is distance between q_i and q_j.

Notice $V(\vec{r}_i) = \frac{1}{4\pi\varepsilon_0}\sum_{j\neq i}\frac{q_j}{r_{ij}}$ is potential due to

$q_1, q_2, \dots, \hat{q}_i, \dots, q_n$ (delete q_i) then

$$\boxed{W = \frac{1}{2}\sum_{i=1}^{n} q_i\left(\frac{1}{4\pi\varepsilon_0}\sum_{j\neq i}\frac{q_j}{r_{ij}}\right) = \frac{1}{2}\sum_{i=1}^{n} q_i V(\vec{r}_i)}$$

Allow $n \longrightarrow \infty$ to consider continuous charge distribution then $\sum_i$ replaced with $\int$ as appropriate

$$\boxed{W = \frac{1}{2}\int \rho V \, d\tau}$$

we'll study this next

To assemble charge distribution we can imagine bringing in q_1, then q_2 then q_3 etc.

1.) q_1 is free, nothing else to repulse it

2.) q_2 : $W_2 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}}$

3.) q_3 : $W_3 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right)$

4.) q_4 : $W_4 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} + \frac{q_1 q_4}{r_{14}} + \frac{q_2 q_4}{r_{24}} + \frac{q_3 q_4}{r_{34}}\right)$

ENERGY TO CREATE CONTINUOUS CHARGE DISTRIBUTION

Since $\nabla \cdot \vec{E} = \rho/\varepsilon_0$ we find $\rho = \varepsilon_0(\nabla \cdot \vec{E})$

$$W = \frac{1}{2}\int \rho V \, d\tau \qquad (dQ = \rho \, d\tau)$$

$$= \frac{\varepsilon_0}{2}\int_{\mathcal{V}} (\nabla \cdot \vec{E}) V \, d\tau$$

$$= \frac{\varepsilon_0}{2}\left[-\int_{\mathcal{V}} \vec{E} \cdot (\nabla V) \, d\tau + \oint_{\partial \mathcal{V}} V \vec{E} \cdot d\vec{a}\right]$$

$$= \frac{\varepsilon_0}{2}\left(\int_{\mathcal{V}} E^2 \, d\tau + \oint_{\partial \mathcal{V}} V \vec{E} \cdot d\vec{a}\right)$$

Given $V(\infty) = 0$ we find $\boxed{W = \frac{\varepsilon_0}{2}\int E^2 \, d\tau}$

E11 Consider spherical shell charge Q at $r = R$, the energy to construct this distribution we can calculate two ways.

$$V(r) = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0}\dfrac{1}{R} & : 0 \leq r \leq R \\ \dfrac{Q}{4\pi\varepsilon_0}\dfrac{1}{r} & : r \geq R \end{cases} \qquad \vec{E} = \begin{cases} 0 & : 0 \leq r \leq R \\ \dfrac{Q}{4\pi\varepsilon_0}\dfrac{\hat{r}}{r^2} & : r \geq R \end{cases}$$

Thus, for surface charge,

$$W = \frac{1}{2}\int_{S_R} \sigma V \, da = \frac{1}{2}\int_{S_R} \frac{Q}{4\pi R^2} \cdot \frac{Q}{4\pi\varepsilon_0 R} \, da = \frac{Q^2}{2(4\pi)^2 \varepsilon_0 R^3} \underbrace{\int_{S_R} da}_{4\pi R^2} = \boxed{\frac{Q^2}{8\pi\varepsilon_0 R}}$$

Alternatively, we can integrate E^2 over all space, since $E = 0$ for $0 \leq r < R$ we simply integrate from R to ∞,

$$W = \frac{\varepsilon_0}{2}\int E^2 \, d\tau = \frac{\varepsilon_0}{2}\int_{\text{space}} \left(\frac{Q}{4\pi\varepsilon_0}\right)^2 \frac{1}{r^4} \cdot r^2 \sin\theta \, d\phi \, d\theta \, dr$$

$$= \frac{4\pi\varepsilon_0 Q^2}{2(4\pi\varepsilon_0)^2}\int_R^\infty \frac{dr}{r^2} = \frac{Q^2}{8\pi\varepsilon_0}\left[\frac{-1}{\infty} + \frac{1}{R}\right] = \boxed{\frac{Q^2}{8\pi\varepsilon_0 R}}$$

CONDUCTORS

A perfect conductor contains an unlimited supply of free charges.
A perfect insulator has no free charge.
When we say "conductor" we mean perfect conductor.

PROPERTIES

(i.) $\vec{E} = 0$ inside a conductor,

(ii.) $\rho = 0$ inside a conductor,

(iii.) any net charge resides on the surface

(iv.) conductor is an equipotential

(v.) $\vec{E} \perp d\vec{a}$ for points near boundary of conductor

The properties follow from the electrostatic condition,

(i.) if $\vec{E} \neq 0$ then free charge moves

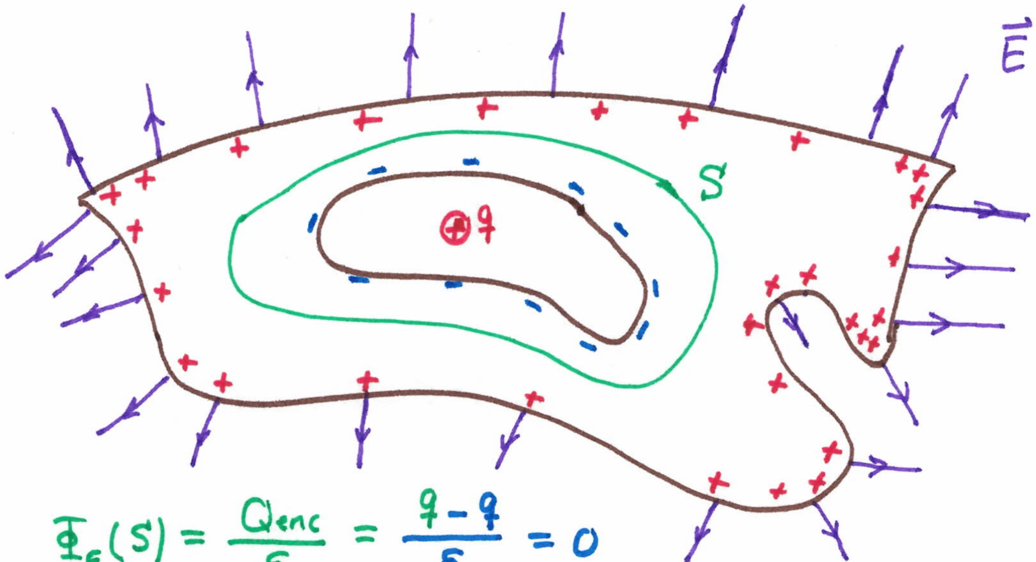
(ii.) if $\rho \neq 0$ then $\nabla \cdot \vec{E} \neq 0$ at some point within conductor and a small bubble would have electric flux and hence $\vec{E} \neq 0$.

(iii.) follows from (ii.)

(iv.) $\vec{E} = -\nabla V = 0 \Rightarrow$ V constant over conductor.

(v.) If $\vec{E}$ had nontrivial tangential component then charge near surface would move
$\therefore$ $\vec{E}$ only has normal component at surface to the conductor.

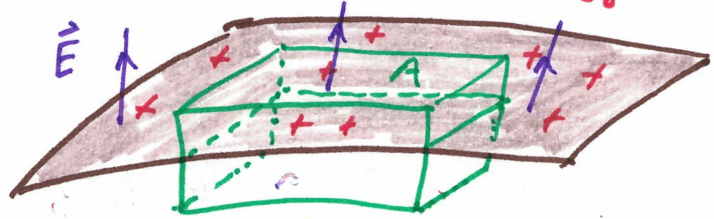
Charge Density and $\vec{E}$-field near conductor



$$\Phi_E(S) = \frac{Q_{enc}}{\varepsilon_0} = \frac{q-q}{\varepsilon_0} = 0$$

$-q$ is <u>induced</u> on interior surface

- If the conductor is nuetral (uncharged) (zero total charge) then $+q$ resides on the outside edge.
- Examining Gaussian thin box as below we see $\Phi_E = AE = \frac{\sigma A}{\varepsilon_0} \Rightarrow E = \frac{\sigma}{\varepsilon_0}$



$$\boxed{\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{n}}$$

Since $\vec{E} = -\nabla V$ we find $\frac{\partial V}{\partial n} = \frac{-\sigma}{\varepsilon_0}$

which means $\boxed{\sigma = -\varepsilon_0 \frac{\partial V}{\partial n}}$ can calculate σ from V at conductor surface.

<u>Remark</u>: see pgs. 100-101 for why the <u>electrostatic</u> pressure $\vec{f} = \frac{1}{2\varepsilon_0}\sigma^2\hat{n}$ force per area or $P = \frac{\varepsilon_0}{2}E^2$

CAPACITORS

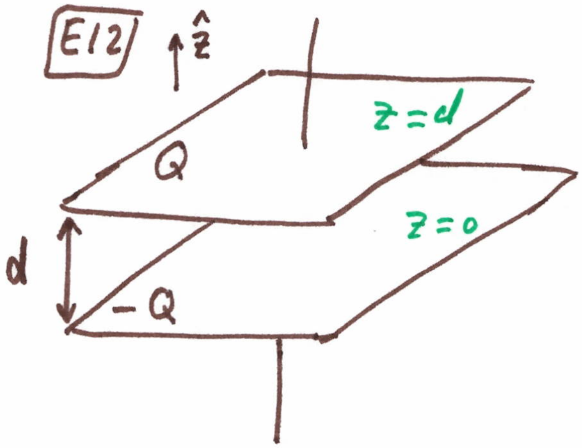
Given two conductors, suppose we place Q on one and $-Q$ on another. If V_+ is voltage on the Q conductor and V_- on the $-Q$ then study

$$V = V_+ - V_- = -\int_{(-)}^{(+)} \vec{E} \cdot d\vec{l}$$

then since $\vec{E}$ is proportional to Q via Coulomb's Law we may define CAPACITANCE of the conductor pair,

$$\boxed{C = \frac{Q}{V}}$$

E12



parallel plates, area A separated distance d

Assume $d \approx 0$ relative to A

$$\vec{E} = \frac{\sigma_+}{2\varepsilon_0}(-\hat{z}) + \frac{\sigma_-}{2\varepsilon_0}(\hat{z})$$

$$\vec{E} = \frac{-Q}{2A\varepsilon_0}\hat{z} - \frac{Q}{2\varepsilon_0 A}\hat{z}$$

$$\vec{E} = \frac{-Q}{A\varepsilon_0}\hat{z} = -\frac{\partial V}{\partial z}$$

$$\Rightarrow \quad V(z) = \frac{Qz}{A\varepsilon_0}$$

$$V = V(d) - V(0) = \frac{Qd}{A\varepsilon_0}$$

$$\therefore \quad \frac{A\varepsilon_0}{d} = \frac{Q}{V} \quad \Rightarrow \quad \boxed{C = \frac{A\varepsilon_0}{d}}$$

<u>Remark</u>: see pgs. 56–59 of my Physics 232 notes for more if you desire it.