1.3.4 Prove the following using induction.

(a) $2n+1 \leq 2^n$ for $n \geq 3$ ($n \in \mathbb{N}$).
(b) $n^2 \leq 3^n$ for all $n \in \mathbb{N}$. (*Hint:* show first that for all $n \in \mathbb{N}$, $2n \leq n^2+1$. This does not require induction.)
(c) $n^3 \leq 3^n$ for all $n \in \mathbb{N}$. (*Hint:* Check the cases $n = 1$ and $n = 2$ directly and then use induction for $n \geq 3$.)

1.3.5 Given a real number $a \neq 1$, prove that

$$1+a+a^2+\cdots+a^n = \frac{1-a^{n+1}}{1-a} \text{ for all } n \in \mathbb{N}.$$

1.3.6 ⊳ The Fibonacci sequence is defined by

$$a_1 = a_2 = 1 \qquad \text{and} \qquad a_{n+2} = a_{n+1} + a_n \text{ for } n \geq 1.$$

Prove that

$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

1.3.7 Let $a \geq -1$. Prove by induction that

$$(1+a)^n \geq 1+na \text{ for all } n \in \mathbb{N}.$$

1.3.8 ⊳ Let $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$. Use Mathematical Induction to prove the binomial theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k},$$

where $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.

1.4 ORDERED FIELD AXIOMS (LECTURE 6)

In this book, we will start from an axiomatic presentation of the real numbers. That is, we will assume that there exists a set, denoted by $\mathbb{R}$, satisfying the ordered field axioms, stated below, together with the completeness axiom, presented in the next section. In this way we identify the basic properties that characterize the real numbers. After listing the ordered field axioms we derive from them additional familiar properties of the real numbers. We conclude the section with the definition of absolute value of a real number and with several results about it that will be used often later in the text.

We assume the existence of a set $\mathbb{R}$ (the set of real numbers) and two operations $+$ and $\cdot$ (addition and multiplication) assigning to each pair of real numbers x, y, unique real numbers $x+y$ and $x \cdot y$ and satisfying the following properties:

(1a) $(x+y)+z = x+(y+z)$ for all $x,y,z \in \mathbb{R}$. (associative)
(1b) $x+y = y+x$ for all $x,y \in \mathbb{R}$. (commutative)
(1c) There exists a unique element $0 \in \mathbb{R}$ such that $x+0 = x$ for all $x \in \mathbb{R}$.

(1d) For each $x \in \mathbb{R}$, there exists a unique element $-x \in \mathbb{R}$ such that $x + (-x) = 0$.

$\mathbb{Q}$ form field

(2a) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in \mathbb{R}$. Associativity of multiplication.
(2b) $x \cdot y = y \cdot x$ for all $x, y \in \mathbb{R}$. Commutivity of multiplication.
(2c) There exists a unique element $1 \in \mathbb{R}$ such that $1 \neq 0$ and $x \cdot 1 = x$ for all $x \in \mathbb{R}$.
(2d) For each $x \in \mathbb{R} \setminus \{0\}$, there exists a unique element $x^{-1} \in \mathbb{R}$ such that $x \cdot (x^{-1}) = 1$. (We also write $1/x$ instead of x^{-1}.)
(2e) $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in \mathbb{R}$. (distributive property)

$\mathbb{Q}(\sqrt{2}) \subseteq \mathbb{R}$

$a + b\sqrt{2}$

$a, b \in \mathbb{Q}$

We often write xy instead of $x \cdot y$.

In addition to the *algebraic axioms* above, there is a relation $<$ on $\mathbb{R}$ that satisfies the *order axioms* below:

(3a) For all $x, y \in \mathbb{R}$, exactly one of the three relations holds: $x = y$, $y < x$, or $x < y$.
(3b) For all $x, y, z \in \mathbb{R}$, if $x < y$ and $y < z$, then $x < z$. (transitive)
(3c) For all $x, y, z \in \mathbb{R}$, if $x < y$, then $x + z < y + z$.
(3d) For all $x, y, z \in \mathbb{R}$, if $x < y$ and $0 < z$, then $xz < yz$.

We will use the notation $x \leq y$ to mean $x < y$ or $x = y$. We may also use the notation $x > y$ to represent $y < x$ and the notation $x \geq y$ to mean $x > y$ or $x = y$.

A set $\mathbb{F}$ together with two operations $+$ and $\cdot$ and a relation $<$ satisfying the 13 axioms above is called an *ordered field*. Thus, the real numbers are an example of an ordered field. Another example of an ordered field is the set of rational numbers $\mathbb{Q}$ with the familiar operations and order. The integers $\mathbb{Z}$ do not form a field since for an integer m other than 1 or -1, its reciprocal $1/m$ is not an integer and, thus, axiom 2(d) above does not hold. In particular, the set of positive integers $\mathbb{N}$ does not form a field either. As mentioned above the real numbers $\mathbb{R}$ will be defined as the ordered field which satisfies one additional property described in the next section: the completeness axiom.

From these axioms, many familiar properties of $\mathbb{R}$ can be derived. Some examples are given in the next proposition. The proof illustrates how the given axioms are used at each step of the derivation.

Proposition 1.4.1 For $x, y, z \in \mathbb{R}$, the following hold:

(a) If $x + y = x + z$, then $y = z$;
(b) $-(-x) = x$;
(c) If $x \neq 0$ and $xy = xz$, then $y = z$;
(d) If $x \neq 0$, then $1/(1/x) = x$;
(e) $0x = 0 = x0$;
(f) $-x = (-1)x$;
(g) $x(-z) = (-x)z = -(xz)$.
(h) If $x > 0$, then $-x < 0$; if $x < 0$, then $-x > 0$;
(i) If $x < y$ and $z < 0$, then $xz > yz$;
(j) $0 < 1$.

Proof: (a) Suppose $x+y=x+z$. Adding $-x$ (which exists by axiom (2d)) to both sides, we have

$$(-x)+(x+y)=(-x)+(x+z).$$

Then axiom (1a) gives

$$[(-x)+x]+y=[(-x)+x]+z.$$

Thus, again by axiom (2d), $0+y=0+z$ and, by axiom (1c), $y=z$. (completes proof of Prop 1.4.1a)

(b) Since $(-x)+x=0$, we have (by uniqueness in axiom (2d)) $-(-x)=x$.

The proofs of (c) and (d) are similar.

(e) Using axiom (2e) we have $0x=(0+0)x=0x+0x$. Adding $-(0x)$ to both sides (axiom (2d)) and using axioms (1a) and (1c), we get

$$0=-(0x)+0x=-(0x)+(0x+0x)=(-(0x)+0x)+0x=0+0x=0x.$$

That $0x=x0$ follows from axiom (2b).

(f) Using axioms (2c) and (2e) we get $x+(-1)x=1x+(-1)x=(1+(-1))x$. From axiom (2d) we get $1+(-1)=0$ and from part (e) we get $x+(-1)x=0x=0$. From the uniqueness in axiom (2d) we get $(-1)x=-x$ as desired.

(g) Using axioms (2e) and (1c) we have $xz+x(-z)=x(z+(-z))=x0=0$. Thus, using axiom (2d) we get that $x(-z)=-(xz)$. The other equality follows similarly.

(h) From $x>0$, using axioms (3c) and (1c) we have $x+(-x)>0+(-x)=-x$. Thus, using axiom (2d), we get $0>-x$. The other case follows in a similar way.

$x<y,\ z<0 \Rightarrow zx>yz$

? (i) Since $z<0$, by part (h), $-z>0$. Then, by axiom (3d), $x(-z)<y(-z)$. Combining this with part (g) we get $-xz<-yz$. Adding $xz+yz$ to both sides and using axioms (1a), (3c), (1b), and (1c) we get

$$xy=(-xz+xz)+xy=-xz+(xz+xy)<-xy+(xz+xy)=-xy+(xy+xz)=(-xy+xy)+xz=xz.$$

(j) Axiom (2c) gives that $1\neq 0$. Suppose, by way of contradiction, that $1<0$. Then by part (i), $1\cdot 1>0\cdot 1$. Since $1\cdot 1=1$, by axiom (2c) and $0\cdot 1=0$ by part (e), we get $1>0$ which is a contradiction. It follows that $1>0$. □

Note that we can assume that the set of all natural numbers is a subset of $\mathbb{R}$ (and of any ordered field, in fact) by identifying the 1 in $\mathbb{N}$ with the 1 in axiom (2c) above, the number 2 with $1+1$, 3 with $1+1+1$, etc. Furthermore, since $0<1$ (from part (j) of the previous proposition), axiom (3c) gives, $1<2<3$, etc (in particular all these numbers are distinct). In a similar way, can include $\mathbb{Z}$ and $\mathbb{Q}$ as subsets.

We say that a real number x is *irrational* if $x\in\mathbb{R}\setminus\mathbb{Q}$, that is, if it is not rational.

Definition 1.4.1 Given $x\in\mathbb{R}$, define the *absolute value* of x by

$$|x|=\begin{cases} x, & \text{if } x\geq 0;\\ -x, & \text{if } x<0.\end{cases}$$

Suppose $x < y$ and $z < 0$.

Then $-z > 0$ from Prop. 1.4.1h.

So, $(-z)x < (-z)y$ by Axiom 3d

$$-xz < -yz$$

Add xz by Axiom 3c,

$$-xz + xz < -yz + xz \rightarrow \cancel{0 < xz - yz}$$

$$0 < -yz + xz \quad \text{by Axiom 1d}$$

Add yz by Axiom 3c,

$$0 + yz < (-yz + xz) + yz$$

$$\underline{yz < xz}.$$

associativity and ~~Axiom~~ commutativity Axiom 1d

$$x<y,\ z<0 \Rightarrow yz < xz \Leftrightarrow xz > yz.$$

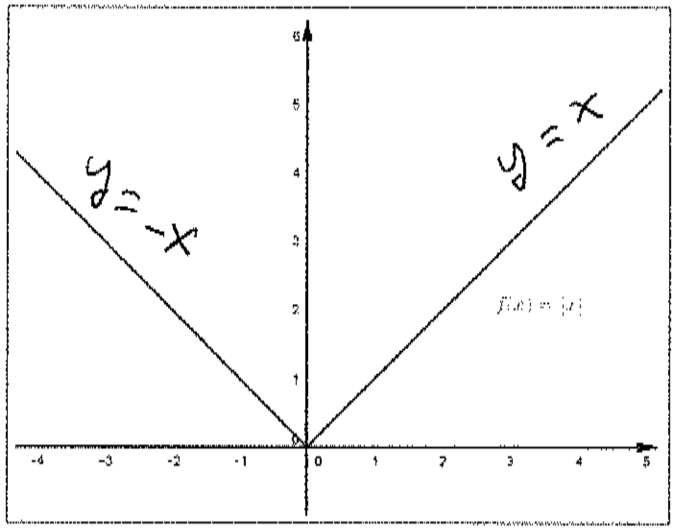


Figure 1.1: The absolute value function.

$$y = \sqrt{x^2} = \begin{cases} x : x \geq 0 \\ -x : x < 0 \end{cases}$$

The following properties of absolute value follow directly from the definition.

Proposition 1.4.2 Let $x, y, M \in \mathbb{R}$ and suppose $M > 0$. The following properties hold:

(a) $|x| \geq 0$;

(b) $|-x| = |x|$;

(c) $|xy| = |x||y|$;

(d) $|x| < M$ if and only if $-M < x < M$. (The same holds if $<$ is replaced with $\leq$.)

Proof: We prove (d) and leave the other parts as an exercise.

(d) Suppose $|x| < M$. In particular, this implies $M > 0$. We consider the two cases separately: $x \geq 0$ and $x < 0$. Suppose first $x \geq 0$. Then $|x| = x$ and, hence, $-M < 0 \leq x = |x| < M$. Now suppose $x < 0$. Then $|x| = -x$. Therefore, $-x < M$ and, so $x > -M$. It follows that $-M < x < 0 < M$.

For the converse, suppose $-M < x < M$. Again, we consider different cases. If $x \geq 0$, then $|x| = x < M$ as desired. Next suppose $x < 0$. Now, $-M < x$ implies $M > -x$. Then $|x| = -x < M$. $\square$

Note that as a consequence of part (d) above, since $|x| \leq |x|$ we get $-|x| \leq x \leq |x|$.

The next theorem will play an important role in the study of limits.

immediate consequence of Prop. 1.4.2 d.

$|x| \leq |x| = M$

Theorem 1.4.3 — Triangle Inequality. Given $x, y \in \mathbb{R}$,

$$|x+y| \leq |x| + |y|.$$

Proof: From the observation above, we have

$$-|x| \leq x \leq |x|$$

$$-|y| \leq y \leq |y|.$$

Adding up the inequalities gives

$$-|x| - |y| \leq x + y \leq |x| + |y|. \quad \Longrightarrow \quad -(|x|+|y|) \leq x+y \leq |x|+|y|$$

$$|x+y| \leq |x|+|y|$$

Prop 1.4.2 d) If $M > 0$ and $x \in \mathbb{R}$ then $|x| < M \Leftrightarrow -M < x < M$.

Proof) $\Rightarrow$) Assume $|x| < M$. Consider 2 cases,

(1.) $x \geq 0$ then $|x| = x$ and $|x| < M \Rightarrow x < M$

But $M > 0$ and so $-M < 0 \leq x < M \Rightarrow -M < x < M$.

(2.) $x < 0$ then $|x| = -x$ and $|x| < M \Rightarrow -x < M$

thus $-M < x < 0 < M$ $\therefore$ $-M < x < M$.

$\Leftarrow$) Conversely suppose $-M < x < M$.

(1.) If $x \geq 0$ then $|x| = x$ and so $-M < |x| < M \Rightarrow |x| < M$.

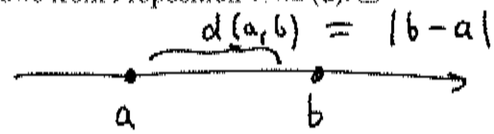
(2.) If $x < 0$ then $|x| = -x$ and so $-M < x < M \Rightarrow -x < M$

which shows $|x| < M$. //

Since $-|x| - |y| = -(|x| + |y|)$, the conclusion follows from Proposition 1.4.2 (d). $\square$

Corollary 1.4.4 For any $x, y \in \mathbb{R}$,

$$||x| - |y|| \leq |x - y|.$$



Remark 1.4.5 The absolute value has a geometric interpretation when considering the numbers in an ordered field as points on a line. The number $|a|$ denotes the distance from the number a to 0. More generally, the number $d(a,b) = |a - b|$ is the distance between the points a and b. It follows easily from Proposition 1.4.2 that $d(x,y) \geq 0$, and $d(x,y) = 0$ if and only if $x = y$. Moreover, the triangle inequality implies that

$$d(x,y) \leq d(x,z) + d(z,y),$$

for all numbers x, y, z.

Exercises

1.4.1 Prove that n is an even integer if and only if n^2 is an even integer. (*Hint:* prove the "if" part by contraposition, that is, prove that if n is odd, then n^2 is odd.)

1.4.2 Prove parts (c) and (d) of Proposition 1.4.1

1.4.3 Let $a, b, c, d \in \mathbb{R}$. Suppose $0 < a < b$ and $0 < c < d$. Prove that $ac < bd$.

1.4.4 Prove parts (a), (b), and (c) of Proposition 1.4.2.

1.4.5 ▶ Prove Corollary 1.4.4.

1.4.6 Given two real numbers x and y, prove that

$$\max\{x,y\} = \frac{x + y + |x - y|}{2} \text{ and } \min\{x,y\} = \frac{x + y - |x - y|}{2}.$$

1.4.7 Let $x, y, M \in \mathbb{R}$. Prove the following

(a) $|x|^2 = x^2$.

(b) $|x| < M$ if and only if $x < M$ and $-x < M$.

(c) $|x + y| = |x| + |y|$ if and only if $xy \geq 0$.

1.5 THE COMPLETENESS AXIOM FOR THE REAL NUMBERS

There are many examples of ordered fields. However, we are interested in the field of real numbers. There is an additional axiom that will distinguish this ordered field from all others. In order to introduce our last axiom for the real numbers, we first need some definitions.

Definition 1.5.1 Let A be a subset of $\mathbb{R}$. A number M is called an *upper bound* of A if

$$x \leq M \text{ for all } x \in A.$$

If A has an upper bound, then A is said to be *bounded above*.

LECTURE 6: ORDERED FIELD AXIOMS (§1.4 in text)

Proof for Corollary 1.4.4 on pg 22 (at nearly end of video)

$|x| = |x - y + y| \leq |x - y| + |y|$

$\therefore \quad |x| - |y| \leq |x - y|$

$|-a| = |a|$

Symmetrically,

$|y| = |y - x + x| \leq |y - x| + |x|$

$|y - x| = |-(x-y)|$
$= |x-y|$

$\therefore \quad |y| - |x| \leq |x - y|$

But, $\big||x| - |y|\big| = \pm(|x| - |y|)$

Hence, $\big||x| - |y|\big| \leq |x - y|$.