LECTURE 6 : CONTINUOUS MAPS

Def^n/ Let X, Y be topological spaces and $f: X \to Y$ a function then f is <u>continuous</u> if $f^{-1}(A) = \{x \in X \mid f(x) \in A\}$ is open in X for any open A in Y. That is, the <u>inverse image</u> of open sets is open for a continuous map. a.k.a "preimage"

Th^m/ 1.) $f: X \to Y$ is continuous iff inverse image of all closed sets is closed
2.) $f: X \to Y$ is continuous iff inverse image of any $B \in \mathcal{B}$ is open ($\mathcal{B}$ is top. basis of Y)

<u>Proof</u> (1.) $\boxed{\Rightarrow}$ Suppose f continuous and $C \subseteq Y$ is closed. Then $Y - C$ is open in Y thus $f^{-1}(Y - C) = X - f^{-1}(C)$ is open in X by cont. of f. Hence $f^{-1}(C)$ is closed in X.

$\boxed{\Leftarrow}$ <u>Suppose C closed in $Y \Rightarrow f^{-1}(C)$ closed in X.</u>$_*$ Let $A \subseteq Y$ be open then $Y - A$ is closed since $Y - (Y - A) = A$ is open. Then $f^{-1}(Y - A) = X - f^{-1}(A)$ is closed by assumption $*$ hence $X - (X - f^{-1}(A)) = f^{-1}(A)$ is open $\therefore f$ continuous.

(2.) follows from the identity $f^{-1}\left(\bigcup_i A_i\right) = \bigcup_i f^{-1}(A_i)$. //

Lemma (3.25) Let $f : X \to Y$ be a map between topological spaces then f is continuous iff $f(\bar{A}) \subseteq \overline{f(A)}$ for any subset $A \subseteq X$

Proof: $\Rightarrow$ Suppose f continuous and let $A \subseteq X$. Then $\overline{f(A)}$ is closed hence $f^{-1}(\overline{f(A)})$ is closed and contains A. Thus $\bar{A} \subset f^{-1}(\overline{f(A)})$

$$\Rightarrow f(\bar{A}) \subset \overline{f(A)}.$$

$\Leftarrow$ Suppose $f(\bar{A}) \subseteq \overline{f(A)}$ for any subset $A \subseteq X$. Let $C \subseteq Y$ be closed. Then $\bar{C} = C$. Let $A = f^{-1}(C)$ then

$$f\left(\overline{f^{-1}(C)}\right) \subset \overline{f(f^{-1}(C))} = \bar{C} = C$$

thus $\overline{f^{-1}(C)} \subseteq f^{-1}(C)$ yet $f^{-1}(C) \subseteq \overline{f^{-1}(C)}$ $\therefore$ $\overline{f^{-1}(C)} = f^{-1}(C)$ and we conclude f is continuous. //

Remark: $f(\bar{A}) \subseteq \overline{f(A)}$ means the points which adhere to A map into a subset of the points which adhere to $f(A)$ (like Def^{n} 1.4 in the introductory chapter)

Thm/ If $f : X \to Y$ and $g : Y \to Z$ are continuous maps on topological spaces then $g \circ f : X \to Z$ is continuous

<u>Proof</u>: Let $g \circ f : X \to Z$ be defined by $(g \circ f)(x) = g(f(x))$ $\forall x \in X$ and suppose $U \subseteq Z$ is open. Suppose f, g continuous thus $g^{-1}(U) \subseteq Y$ is an open set by continuity of g. But, then $f^{-1}(g^{-1}(U))$ is an open set by cont. of f. Notice that $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ thus $g \circ f$ is continuous as we've shown the inverse image of open is open under $g \circ f$. //

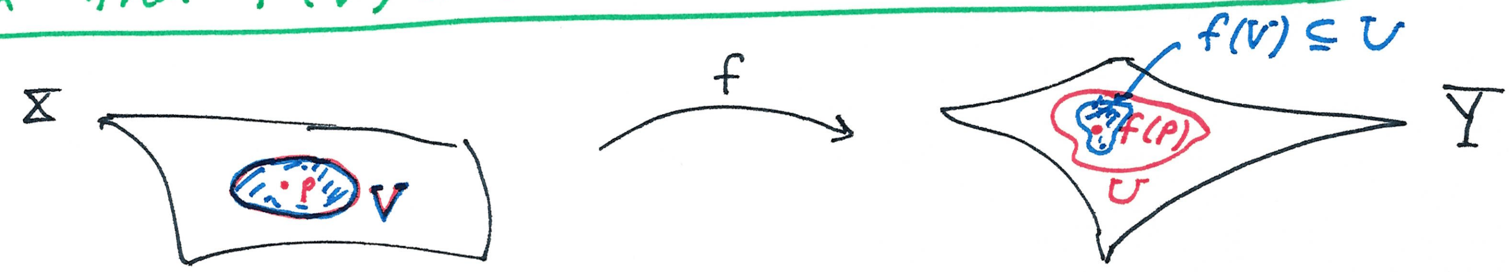
<u>Why $(g \circ f)^{-1})(U) = f^{-1}(g^{-1}(U))$</u>

Let $x \in (g \circ f)^{-1}(U)$ then $(g \circ f)(x) \in U \Rightarrow g(f(x)) \in U$ thus $f(x) \in g^{-1}(U)$ and so $x \in f^{-1}(g^{-1}(U))$ $\therefore$ $(g \circ f)^{-1}(U) \subseteq f^{-1}(g^{-1}(U))$.
Likewise, if $x \in f^{-1}(g^{-1}(U))$ then $f(x) \in g^{-1}(U)$ hence $g(f(x)) \in U$ and so $(g \circ f)(x) \in U$ which shows $x \in (g \circ f)^{-1}(U)$ $\therefore$ $f^{-1}(g^{-1}(U)) \subseteq (g \circ f)^{-1}(U)$.

Consequently, $\underline{(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))}$.

<u>Remark</u>: none of this requires f^{-1} or g^{-1} be functions.

Def^{n}/ A map $f: X \to Y$ is continuous at a <u>point</u> $p \in X$ if for any nbhd U of $f(p)$ there is a nbhd V of p such that $f(V) \subseteq U$



Th^{m}/ A map $f: X \to Y$ is continuous iff f is continuous at every pt. of X

<u>Proof</u>: $\Rightarrow$] Suppose f is continuous and let $p \in X$ and suppose U is nbhd of $f(p)$ in Y. By def^{n} of nbhd $\exists$ open $A \subseteq Y$ for which $f(p) \in A \subseteq U$. Note $V = f^{-1}(A)$ is open in X by cont. of f and V serves as nbhd of p with $f(V) \subseteq U$.
$\therefore$ f continuous at arbitrary $p \in X$.

$\Leftarrow$] Suppose f continuous at every pt. in X. Let $A \subseteq Y$ be an open set in Y. Let $x \in f^{-1}(A)$ then A is nbhd of $f(x)$ hence $\exists\, V$ a nbhd of x for which $f(V) \subseteq A$. Then $V \subseteq f^{-1}(A)$ and we find $f^{-1}(A)$ is nbhd of x. But, x was arbitrary and we conclude $f^{-1}(A)$ is open $\therefore$ f continuous. //

Def^{n}/ A <u>homeomorphism</u> is a continuous bijection with continuous inverse. That is, a continuous map $f : X \to Y$ is a homeomorphism if $\exists$ continuous map $g : Y \to X$ such that $f \circ g = id_Y$ and $g \circ f = id_X$. If $\exists$ a homeomorphism $f : X \to Y$ then X is homeomorphic to Y

Def^{n}/ $f : X \to Y$ is called
1.) <u>open</u> if A open $\Rightarrow f(A)$ open
2.) <u>closed</u> if C closed $\Rightarrow f(C)$ closed

<u>Lemma</u>: Let $f : X \to Y$ be continuous map. TFAE
(1) f homeomorphism
(2) f is closed and bijective
(3) f is open and bijective

<u>Proof</u>: <u>(1) $\Rightarrow$ (2.)</u> Let $f : X \to Y$ be a homeomorphism. Then f is bijective. Let $C \subseteq X$ be closed and notice

$$f(C) = (f^{-1})^{-1}(C) \text{ is closed since } f^{-1} \text{ is continuous.}$$

Thus f is closed.

<u>(2.) $\Rightarrow$ (3.)</u> Let f be closed and bijective. We wish to show f open map. Let $A \subseteq X$ be open then $C = X - A$ is closed as $X - C = A$. Thus $f(C)$ is closed hence $Y - f(C)$ is open. Yet f a bijection gives $f(A) = f(X - C) = Y - f(C) \Rightarrow f(A)$ open $\therefore f$ open.

<u>(3) $\Rightarrow$ (1)</u> f open and bijective and $g = f^{-1}$. If $B \subseteq X$ is open then $g^{-1}(B) = f(B)$ is open $\therefore g$ is continuous $\Rightarrow f$ is homeomorphism. //

Ex. 3.26 Let $f : X \to Y$ be ~~continuous map~~ open map and $D \subseteq Y$ a dense subset. Prove $f^{-1}(D)$ is dense in X

D dense means $D \cap V \neq \emptyset$ forall $\emptyset \neq V \in \mathcal{T}_Y$

Let $U \subseteq X$, $U \neq \emptyset$ then $f(U) \neq \emptyset$ and $f(U) \subseteq Y$

thus $f(U) \cap D \neq \emptyset$ by density of D.

Therefore, $\exists\, f(x_0) \in f(U) \cap D$ for some $\underline{x_0 \in U}$ *

$\therefore\ f(x_0) \in f(U)$ <u>and</u> $f(x_0) \in D$

$\underline{x_0 \in U}$ * and $x_0 \in f^{-1}(D)$

$$\Rightarrow f^{-1}(D) \cap U \neq \emptyset$$

$\therefore\ f^{-1}(D)$ is dense in X. //