LECTURE 7 : METRIC SPACES

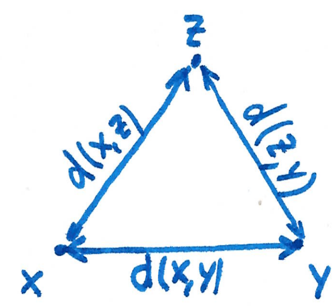
(Manetti §3.4, but I add some here)

Def^n/ A <u>distance</u> on a set X is a function $d: X \times X \to \mathbb{R}$ s.t.

(1.) $d(x,y) \geq 0$ $\forall x,y \in X$, and (non-negative)
$d(x,y) = 0 \iff x = y$.

(2.) $d(x,y) = d(y,x)$ $\forall x,y \in X$ (SYMMETRIC)

(3.) $d(x,y) \leq d(x,z) + d(z,y)$ $\forall x,y,z \in X$ (TRIANGLE INEQ.)



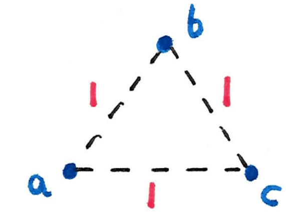
$\boxed{E1}$ Let X be a set, define $d: X \times X \to \mathbb{R}$ by $d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

(1.) Clearly $d(x,x) = 0$ and $d(x,y) = 0 \Rightarrow x = y$ and $d(x,y) \geq 0$ thus (1.) holds.
(2.) $d(x,y) = d(y,x)$ $\forall x,y \in X$ and (3.) holds, just think through the cases.

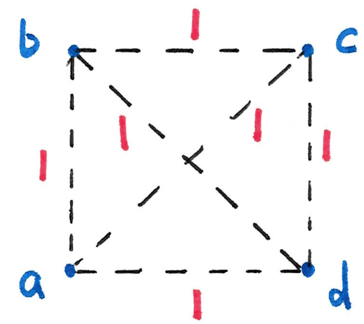
Def^n/ A <u>metric space</u> is a pair (X, d) where X is a set with distance function d.

<u>Remark</u>: $\boxed{E1}$ applies to <u>any</u> set $\therefore$ any set can be made into a metric space ... however, perhaps not in an interesting way...

$\boxed{E2}$ Let $X = \{a, b, c\}$ with $d(a,b) = d(a,c) = d(b,c) = 1$ and d symmetric with $d(a,a) = d(b,b) = d(c,c) = 0$.



$\boxed{E3}$ $X = \{a, b, c, d\}$ with $\boxed{E1}$ distance function



$d(a,b) = d(a,c) = d(a,d) = 1$

$\boxed{E4}$ $X = \mathbb{R}$ with $d(x,y) = |y-x| = \sqrt{(y-x)^2}$ is <u>Euclidean</u> <u>Distance</u> on $\mathbb{R}$.

$\boxed{E5}$ $X = \mathbb{R}^n$ with $d(x,y) = \sqrt{(y_1 - x_1)^2 + \cdots + (y_n - x_n)^2}$ is <u>Euclidean Distance on $\mathbb{R}^n$</u>

$\boxed{E6}$ $X = \mathbb{C}^n$ with $d(x,y) = \sqrt{|y_1 - x_1|^2 + \cdots + |y_n - x_n|^2}$ is <u>Euclidean Distance</u> on $\mathbb{C}^n$

Here $|y-x|^2 = (y-x)\overline{(y-x)} = (\mathrm{Re}(y-x))^2 + (\mathrm{Im}(y-x))^2 \quad \forall x, y \in \mathbb{C}$.

Def^{n}/ If V is a <u>vector</u> <u>space</u> over $\mathbb{R}$ then $\|\cdot\| : V \to \mathbb{R}$ is a <u>norm</u> on V provided

or <u>length</u>

(1.) $\|x\| \geq 0 \quad \forall x \in V$

(2.) $\|x\| = 0 \iff x = 0$

(3.) $\|cx\| = |c|\|x\| \quad \forall c \in \mathbb{R}, x \in V$

(4.) $\|x+y\| \leq \|x\| + \|y\| \quad \forall x, y \in V$

We call $(V, \|\cdot\|)$ a <u>normed</u> <u>linear</u> <u>space</u> or NLS.

$\boxed{E7}$ Given an NLS $(V, \|\cdot\|)$ we may define the distance induced by norm $\|\cdot\|$ as follows:

$$d(x,y) = \|y - x\|$$

For $\mathbb{R}^n$ the Euclidean norm $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$ thus $\boxed{E5}$ is an example of distance function induced from a norm.

Remark: in contrast $\mathbb{X} = \{a,b,c,d\}$ or $\mathbb{X} = \{a,b,c\}$ of $\boxed{E3}$ & $\boxed{E2}$ respective are not even vector spaces, so they're not NLS... not every distance function is induced from some norm, just a special subset which appear often in applications.

$\boxed{E8}$ $V = \mathbb{R}^{n\times n}$ then $\|A\| = \sqrt{\operatorname{trace}(A^TA)} = \sqrt{A_{11}^2 + A_{12}^2 + \cdots + A_{nn}^2}$ is the so-called Frobenius norm on $\mathbb{R}^{n\times n}$. We may define the Euclidean distance between $A, B \in \mathbb{R}^{n\times n}$ by $d(A,B) = \|B - A\|$. It's unsurprising how it works:

$$d\Bigg(\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{A}, \underbrace{\begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}}_{B}\Bigg) = \left\| \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right\| \qquad d_{E1}(A,B) = 1$$

$$= \left\| \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \right\|$$

$$= \sqrt{4^2 + 3^2 + 2^2 + 1^2}$$

$$= \sqrt{30}.$$

Remark: there is more to say about inner product spaces. I'll digress here a bit...

Competing Concepts of Distance for $\mathbb{R}^n$

$$d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$$

$$d_p(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p}$$ ($p = 2$ is Euclidean Distance) ($d_2 = d$)

$$d_\infty(x,y) = \max_i \{ |x_i - y_i| \}$$

Def$^{=}$/ The <u>CIRCLE</u> of radius R centered at x_0 for metric space (X, d) is the set of all pts. distance R from x_0; $\underbrace{\{x \in X \mid d(x, x_0) = R\}}_{C_d(x_0; R)}$

$\boxed{E9}$ Consider $X = \mathbb{R}^2$ let's study $C_d(0,1)$

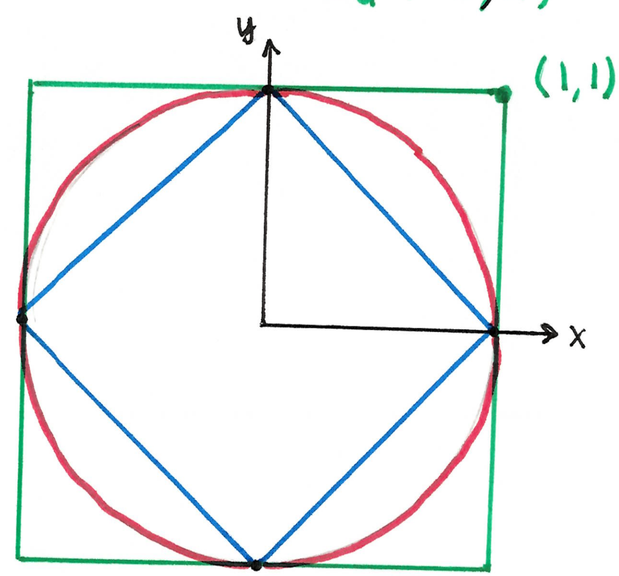
$C_d(0;1) = \{(x,y) \mid d((x,y),(0,0)) = 1\}$

$C_{d_1}(0;1) = \{(x,y) \mid |x| + |y| = 1\}$

$C_{d_2}(0;1) = \{(x,y) \mid \sqrt{|x|^2 + |y|^2} = 1\}$

$\vdots$

$C_{d_\infty}(0;1) = \{(x,y) \mid \max\{|x|, |y|\} = 1\}$



Comparing Distances $d_\infty(x,y) \le d(x,y) \le d_1(x,y) \le n \cdot d_\infty(x,y)$

for instance with $n = 2$ note $\underbrace{d_\infty((1,1),0)}_{1} \le \underbrace{d((1,1),0)}_{\sqrt{2}} \le \underbrace{d_1((1,1),0)}_{2} \le 2$

$\boxed{E10}$ <u>STANDARD</u> <u>BOUND</u>: given metric space (X, d) define

$\bar{d} : X \times X \longrightarrow \mathbb{R}$ by $\bar{d}(x,y) = \min(1, d(x,y))$ $\forall x, y \in X$.

$$\bar{d}(x,x) = \min(1, d(x,x)) = \min(1,0) = 0.$$

$\bar{d}(x,y) = \min(1, d(x,y)) \geq 0$ since $d(x,y) \geq 0$ is given.

$\bar{d}(x,y) = \min(1, d(x,y)) = \min(1, d(y,x)) = \bar{d}(y,x)$ since $d(x,y) = d(y,x)$.

The triangle-inequality remains, let $x, y, z \in X$ consider $\bar{d}(x,z) + \bar{d}(z,y) \geq 1$ then as $\bar{d}(x,y) \leq 1$ by construction of $\bar{d}$ we have $\bar{d}(x,y) \leq 1 \leq \bar{d}(x,z) + \bar{d}(z,y)$. Thus suppose $\bar{d}(x,z) + \bar{d}(z,y) < 1$ from which we find $\bar{d}(x,z), \bar{d}(z,y) < 1 \Rightarrow \bar{d}(x,z) = d(x,z)$ & $\bar{d}(z,y) = d(z,y)$ hence $d(x,z) + d(z,y) \geq d(x,y)$ and thus,

$$\bar{d}(x,y) \leq d(x,y) \leq d(x,z) + d(z,y) = \bar{d}(x,z) + \bar{d}(z,y).$$

Defn/ If (X, d) is metric space then subset $A \subseteq X$ is <u>bounded</u> if $\exists M \in \mathbb{R}$ such that $d(a,b) \leq M$ $\forall a, b \in A$. Also, a map $f : Z \longrightarrow X$ is <u>bounded</u> if $f(Z)$ is bounded.

<u>Remark</u>: $\mathbb{R}^2$ is not bounded by Euclidean metric however, for $\bar{d} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ note $\bar{d}(x,y) \leq 1$ $\forall x, y \in \mathbb{R}^2$ thus $\mathbb{R}^2$ is <u>bounded</u> in the metric space $(\mathbb{R}^2, \bar{d})$.

METRIC TOPOLOGY: A.K.A. TOPOLOGY INDUCED FROM A METRIC

Def/ Let (X, d) be a metric space then

$$B(x,r) = \{ y \in X \mid d(x,y) < r \}$$

is the <u>open</u> <u>ball</u> centered at x with radius r (with respect to distance d)

Th/Def (Topology induced by distance function) Let (X, d) be metric space. The topology induced by d is called the <u>metric</u> <u>topology</u>. <u>In particular, $A \subseteq X$ is defined to be open in the metric topology if for any $x \in A$ there exists $r > 0$ such that $B(x,r) \subseteq A$.</u>

Proof: ① $A = \emptyset$ is open by default. Observe $d(x,x) = 0$ for each $x \in X$. If $r > 0$ then $d(x,x) = 0 < r$ thus $x \in B(x,r)$. Observe $x \in B(x,1)$ $\forall x \in X$ thus $X = \bigcup_{x \in X} B(x,1)$ (this shows X is union of open balls)

not needed for Th^m But, worthy of note.

② Ok, let $x_o \in X$ then $B(x_o, 1) \subseteq X$ $\therefore$ X is open in metric topology.

③ Suppose $\{A_i\}$ is collection of open sets and let $x \in \bigcup_i A_i$ then $\exists j$ for which $x \in A_j \Rightarrow \exists r > 0$ with $B(x,r) \subseteq A_j \subseteq \bigcup_i A_i \Rightarrow \bigcup_i A_i$ is open.

④ If A, B are open and $x \in A \cap B$ then $x \in A$ and $x \in B$, both open hence $\exists r_A, r_B > 0$ s.t. $B(x, r_A) \subseteq A$ & $B(x, r_B) \subseteq B$. Let $r = \min(r_A, r_B)$ then $B(x,r) \subseteq B(x, r_A)$ and $B(x,r) \subseteq B(x, r_B) \Rightarrow B(x,r) \subseteq A \cap B$. //

Remark: $\mathbb{R}^n$, $\mathbb{C}^n$ with topology induced from Euclidean metric are said to carry the Euclidean topology. Typically this is the default topology used unless an application warrants a different choice of topology.

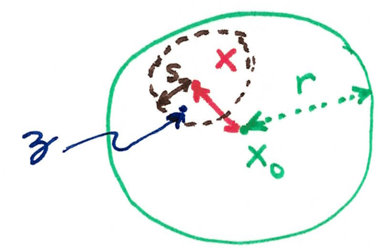
Lemma (3.44) For the topology induced from a distance on X,

(1.) open balls are open.

(2.) $A \subseteq X$ is open iff A is union of open balls. (that is to say open balls form a topological basis for X)

(3.) $U \subseteq X$ is nbhd of point x iff U contains an open ball centered at x iff $\exists r > 0$ s.t. $B(x,r) \subseteq U$.

Proof:



$d(x, x_o) < r$

(1.) let $x \in B(x_o, r)$ where $r > 0$.

Set $s = r - d(x, x_o) > 0$. We wish to show $B(x,s) \subseteq B(x_o, r)$ which then proves (1.).

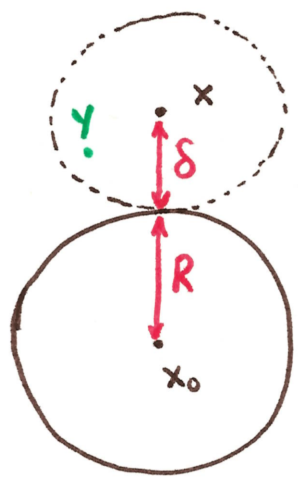
Let $z \in B(x,s)$ and note $d(x_o, z) \leq d(x_o, x) + d(x, z) < d(x_o, x) + s = r$

thus $d(x_o, z) < r \Rightarrow z \in B(x_o, r) \therefore B(x,s) \subseteq B(x_o, r)$.

(2.) If $A \subseteq X$ is open then for each $x \in A$ there exits $r(x) \in \mathbb{R}$ s.t. $B(x, r(x)) \subseteq A$ then $\bigcup_{x \in A} \{x\} = A$ and as $x \in B(x, r(x))$ $\forall x$ note as $B(x, r(x)) \subseteq A$ we have $\bigcup_{x \in A} B(x, r(x)) \subseteq A$ and it follows $A = \bigcup_{x \in A} B(x, r(x))$.

The converce direction is clear since A a union of open balls means each point in A is contained in an open ball.

Th^m/ Let (X, d) be metric space then $C = \{x' \in X \mid d(x_0, x') \le R\}$ is a closed set in the metric topology for any $x_0 \in X$ and $R \geq 0$



<u>Proof</u>: Let $x_0 \in X$ and suppose $R > 0$. Consider $x \in X - C$. Set $\delta = d(x, x_0) - R$ and note $\delta > 0$ since $x \notin C \Rightarrow d(x, x_0) > R$. Suppose $y \in B(x, \delta)$ then by triangle inequality,

$$d(x_0, x) \le d(x_0, y) + d(y, x)$$

Thus,

$$\begin{aligned} d(x_0, y) &\ge d(x_0, x) - d(x, y) \\ &= \delta + R - d(x, y) \\ &> \delta + R - \delta \\ &= R \end{aligned}$$

$y \in B(x, \delta)$
$d(x, y) < \delta$
$-d(x, y) > -\delta$

Thus $d(x_0, y) > R$ and so $y \notin C$ which means $y \in X - C$ $\therefore$ $B(x, \delta) \subseteq X - C$

Therefore $X - C$ is open which proves C closed.//

$\boxed{E11}$ Consider the $d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases} \quad \forall x, y \in X.$

$$C = \{ x' \in X \mid d(x_0, x') \leq 1 \} = X$$

$B(x, 1/2) = \{ y \in X \mid d(x,y) < \frac{1}{2} \} = \{x\}$ ← open set in this $\boxed{E1}$ metric topology

$A \subseteq X$ is closed if $X - A$ is open

Note $X - A = \bigcup_{x \notin A} B(x, 1/2)$ thus A is closed.

Every subset of X is both open & closed in this metric's topology.

Consider $B(x,1)$, the closure $\overline{B(x,1)} \neq C$

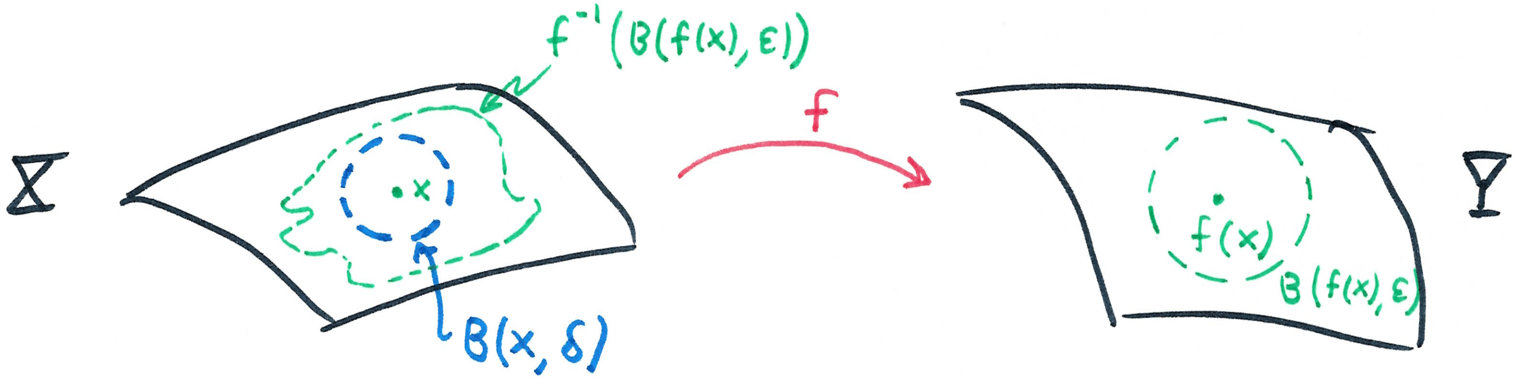
$$\overline{B(x,1)} = \bigcap_{\substack{\text{closed } N \\ \text{sets} \\ \text{containing} \\ B(x,1)}} N = B(x,1) = \{x\}$$

Th$^{\underline{m}}$/ Let $f : (X, d) \to (Y, h)$ be a map between metric spaces and x a point of X. Then f is continuous at x iff for any $\varepsilon > 0$ there exists $\delta > 0$ such that $h(f(x), f(y)) < \varepsilon$ whenever $d(x,y) < \delta$.

<u>Proof</u>: If f is continuous then the inverse image of open sets is open. Let $\varepsilon > 0$ then $B(f(x), \varepsilon) = \{z \in Y \mid h(f(x), z) < \varepsilon\}$ is an open set in Y thus $f^{-1}(B(f(x), \varepsilon)) = \{y \in X \mid f(y) \in B(f(x), \varepsilon)\}$ is open in X. Moreover, $f(x) \in B(f(x), \varepsilon)$ hence $x \in f^{-1}(B(f(x), \varepsilon))$ hence $\exists \delta > 0$ such that $B(x, \delta) \subseteq f^{-1}(B(f(x), \varepsilon))$. Therefore,

$y \in B(x, \delta) \Rightarrow y \in f^{-1}(B(f(x), \varepsilon))$ a.k.a

$d(x,y) < \delta \Rightarrow f(y) \in B(f(x), \varepsilon) \Rightarrow h(f(x), f(y)) < \varepsilon$.



E12

Def^{n}/ Let (X, d) be a metric space. Let $Z \subseteq X$ be a nonempty set, then define <u>distance</u> from Z by

$$d_Z : X \longrightarrow \mathbb{R}, \quad d_Z(x) = \inf_{z \in Z} d(x, z)$$

Observe $d_Z(x) = 0$ for $x \in Z$. ($d(x,x) = 0$ is abs. minimum)

Manetti claims $d_Z(x) = 0 \iff x \in \overline{Z}$. Also,

$$|d_Z(x) - d_Z(y)| \leq d(x, y)$$

To see this, consider we'd like to show:

$$d_Z(y) - d_Z(x) \leq d(x, y)$$

ok, from def^{n} of inf, for any $\varepsilon > 0$, $z \in Z$ such that

$$d_Z(x) + \varepsilon \geq d(x, z)$$

hence,

$$d_Z(y) \leq d(z, y) \leq d(z, x) + d(x, y) \leq d_Z(x) + \varepsilon + d(x, y)$$

Then, $d_Z(y) - d_Z(x) \leq \varepsilon + d(x, y) \Rightarrow d_Z(y) - d_Z(x) \leq d(x, y)$.

<u>Corollary to ε-δ Th^{m}</u>

Let d, h be distances on a set X. The topology induced by d is finer than the topology induced by h iff for any $x \in X$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x,y) < \delta \Rightarrow h(x,y) < \varepsilon$

<u>Proof</u> $\Rightarrow$] Suppose (X, d) generates a topology finer than that induced by (X, h). If U is open in h-topology then $\exists\, V$-open in d-topology with $V \subseteq U$. Consider $B_h(x, \varepsilon) = U$ then $\exists\, V \subseteq B_h(x, \varepsilon)$ where V is open in d-topology hence $\exists\, \delta > 0$ s.t. $B_d(x, \delta) \subseteq V \subseteq B_h(x, \varepsilon)$

$\therefore\ d(x,y) < \delta \Rightarrow h(x,y) < \varepsilon$.

$\Leftarrow$] Suppose for any $x \in X$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x,y) < \delta \Rightarrow h(x,y) < \varepsilon$. Let U be open in h-topology and $x \in U$ then $\exists\, \varepsilon > 0$ s.t. $B_h(x, \varepsilon) \subseteq U$. By assumption $\exists\, \delta > 0$ s.t. $d(x,y) < \delta \Rightarrow h(x,y) < \varepsilon \Rightarrow B_d(x, \delta) \subseteq B_h(x, \varepsilon) \subseteq U$. Hence the d-topology is finer than the h-topology. //

<u>Remark: Manetti's proof</u>: The d-top. is finer than the h-top. iff the identity map $(X, d) \longrightarrow (X, h)$ is continuous.

Def^n/ Two distances on a set X are said to be equivalent if they induce the same topology. A topological space X is metrizable if the topology is induced by a suitable distance.

The problem of metrizability is a well-studied question. Theorems of Tychonoff, Urysohn, Nagata-Smirnov and Bing characterize if it is possible to look at a given topological space as a metric space

Urysohn: every Hausdorff 2^{nd}-countable regular space is metrizable (Urysohn pub. 1925, Tychonoff 1926)

Nagata-Smirnov: X metrizable iff regular, Hausdorff, countably locally finite basis (1950-1951)

Bing: X metrizable iff it is regular and T_0 and has a σ-discrete basis. (1951)

Remark: many examples are metrizable as we've seen. A more exotic example, the group of unitary operators $U(\mathcal{H})$ on a separable Hilbert space $\mathcal{H}$ endowed with the strong operator topology.

Non-examples, un-metrizable spaces:
- "really long line"
- Zariski topology on an algebraic variety or on the spectrum of a ring from algebraic geometry
- Topological vector space of all fncts $\mathbb{R} \to \mathbb{R}$ with topology of pointwise convergence.
- $\mathbb{R}$ with lower limit top.