Lecture 8 : Uniqueness $Th^{\underline{m}}$ & Method of Images

Given a collection of charges on the boundary of some region as well as a finite collection of charges within the region $\mathcal{V}$. The region could extend to infinity, however we focus on case $\rho(\vec{r}) = 0$ for r sufficiently large. Let ρ be the given charge density.

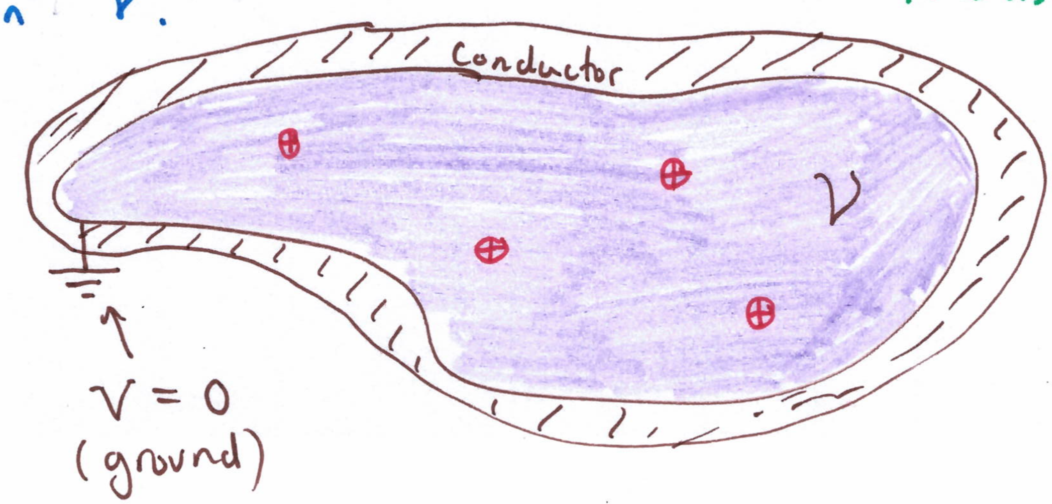
$Th^{\underline{m}}$/ If the charge density ρ is given over some region $\mathcal{V} \subseteq \mathbb{R}^3$ then,

(1.) if given values for V on $\partial\mathcal{V}$ then there is a unique solution to $\nabla^2 V = -\rho/\varepsilon_0$ on $\mathcal{V}$

(2.) if given values for $\frac{\partial V}{\partial n}$ on $\partial\mathcal{V}$ then there is a unique solution to $\nabla^2 V = -\rho/\varepsilon_0$ on $\mathcal{V}$ up to a choice of an additive constant.

(3.) If $V = 0$ is given for <u>some</u> points in $\partial\mathcal{V}$ and $\frac{\partial V}{\partial n}$ is known for others then there is a unique solution to $\underbrace{\nabla^2 V = -\rho/\varepsilon_0}_{\text{Poisson's Equation}}$ on $\mathcal{V}$.

$$\underbrace{\boxed{\nabla^2 V = 0 \text{ wherever } \rho = 0}}_{\text{Laplace's Equation}}$$



Proof of Uniqueness Th$^{\underline{m}}$

Suppose charge density ρ is given within $\mathcal{V}$ with positively oriented boundary $\partial\mathcal{V}$. I'll imagine $\mathcal{V}$ finite, yet large enough to capture <u>all</u> points for which $\rho \neq 0$.

Let V_1 and V_2 be two solutions to Poisson's Eq$^{\underline{n}}$ $\nabla^2 V = -\rho/\varepsilon_0$ and consider $V_3 = V_1 - V_2$,

$$\begin{aligned}
\nabla^2 V_3 &= \nabla \cdot \nabla (V_1 - V_2) \\
&= \nabla \cdot (\nabla V_1 - \nabla V_2) \\
&= \nabla \cdot \nabla V_1 - \nabla \cdot \nabla V_2 \\
&= \nabla^2 V_1 - \nabla^2 V_2 \\
&= -\rho/\varepsilon_0 + \rho/\varepsilon_0 \\
&= 0
\end{aligned}$$

Therefore V_3 solves Laplace's equation $\nabla^2 V = 0$ on $\mathcal{V}$. Consider the following,

$$\nabla \cdot (V \nabla V) = \nabla V \cdot \nabla V + V \nabla \cdot \nabla V$$

$$\therefore \quad \underline{\nabla \cdot (V \nabla V) = \|\nabla V\|^2 + V \nabla^2 V} \quad (*)$$

Using $*$ in what follows as well as $\nabla^2 V_3 = 0$,

$$\int_{\mathcal{V}} \nabla \cdot (V_3 \nabla V_3)\, d\tau = \int_{\mathcal{V}} \big(\|\nabla V_3\|^2 + \underbrace{V_3 \nabla^2 V_3}_{\text{zero since}} \big)\, d\tau$$

$\underbrace{\nabla^2 V_3 = 0}_{V_3 \text{ solves Laplace's Equation.}}$

Then by divergence Th^{m},

$$\boxed{\int_{\partial \mathcal{V}} (V_3 \nabla V_3) \cdot d\vec{a} = \int_{\mathcal{V}} \|\nabla V_3\|^2\, d\tau} \leftarrow **$$

Now the rest of the proof requires applying the boundary conditions given for V_1 & V_2 and hence $V_3 = V_1 - V_2$ as follows:

(1.) Given $V_1(P) = V_2(P)$ for each $P \in \partial \mathcal{V}$ we find $V_3(P) = V_1(P) - V_2(P) = 0$ for each $P \in \partial \mathcal{V}$.

Hence by $**$ we deduce

$$\int_{\mathcal{V}} \|\nabla V_3\|^2\, d\tau = 0$$

It follows V_3 cannot be nonzero within $\mathcal{V}$ and by continuity we find $V_3 \equiv 0$ on $\mathcal{V}$

$$\therefore \quad \underline{V_1 = V_2}.//$$

(2.) Suppose $\frac{\partial V}{\partial n} = (\nabla V) \cdot \hat{n}$ is given on $\partial \mathcal{V}$

then $\frac{\partial V_3}{\partial n} = \frac{\partial V_1}{\partial n} - \frac{\partial V_2}{\partial n} = 0$ on $\partial \mathcal{V}$

then from ** in 2^{nd} line,

$$\int_{\partial \mathcal{V}} (V_3 \nabla V_3) \cdot d\vec{a} = \int_{\mathcal{V}} \| \nabla V_3 \|^2 \, d\tau$$

But, $d\vec{a} = \hat{n} \, da$ thus,

$$\int_{\partial \mathcal{V}} V_3 (\nabla V_3) \cdot \hat{n} \, da = \int_{\partial \mathcal{V}} V_3 \frac{\partial V_3}{\partial n} \, da = 0$$

$$\therefore \int_{\mathcal{V}} \| \nabla V_3 \|^2 d\tau = 0$$

Consequently, $\nabla V_3 \equiv 0$ on $\mathcal{V}$

Thus $V_3 = c \Rightarrow V_1 - V_2 = c \therefore \underline{V_1 = V_2 + c}$.//

(3.) given V or $\frac{\partial V}{\partial n}$ at each point of $\partial \mathcal{V}$

we have $V_3 = 0$ or $\frac{\partial V_3}{\partial n} = 0$ at each

point in $\partial \mathcal{V}$ thus by arguments in (1) or (2)

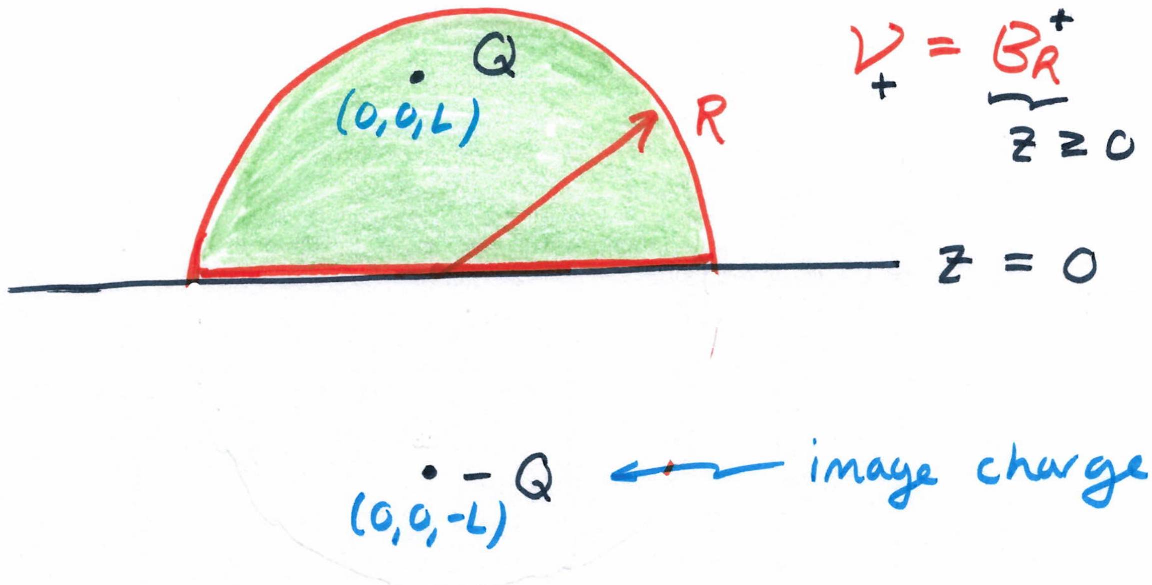
we find $\int_{\mathcal{V}} \| \nabla V_3 \|^2 d\tau = 0 \Rightarrow \nabla V_3 = 0$ on $\mathcal{V}$

Hence, $V_3 = c$, but by $V_3 = 0$ on $\partial \mathcal{V}$

$\therefore V_3 = 0$ and we find $\underline{V_1 = V_2}$.//

METHOD OF IMAGES

E1 Consider the infinite conducting plane at $z=0$. Place a charge Q at $(0,0,L)$. Find the potential and induced surface charge density.



$$\underbrace{V(x,y,z) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{x^2+y^2+(z-L)^2}}\right) - \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{x^2+y^2+(z+L)^2}}\right)}$$

$$\boxed{V(x,y,z) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{x^2+y^2+(z-L)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+L)^2}}\right)}$$

Observe $V(x,y,0) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{x^2+y^2+L^2}} - \frac{1}{\sqrt{x^2+y^2+L^2}}\right) = 0$

Apply uniqueness Th^{m} to $\mathcal{V}_+(R) = \{(x,y,z) \mid r \leq R,\ z \geq 0\}$ to see our image charge based potential gives correct $V(x,y,z)$ in region where $z>0$ and Q induces some surface charge $\sigma(x,y)$ over the conducting plane. Notice $R \to \infty$ gives us this result for the infinite plane.

(next we derive σ_z)

E1 continued

$$V(x,y,z) = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{\underbrace{\sqrt{x^2+y^2+(z-L)^2}}_{d_1}} - \frac{1}{\underbrace{\sqrt{x^2+y^2+(z+L)^2}}_{d_2}}\right]$$

Calculate surface charge density at $z=0$ from

$$\sigma = -\epsilon_0 \left.\frac{\partial V}{\partial n}\right|_{z=0}$$

$$= \frac{-Q}{4\pi}\frac{\partial}{\partial z}\left[\frac{1}{d_1} - \frac{1}{d_2}\right]\Bigg|_{z=0}$$

$$= \frac{-Q}{4\pi}\left[\frac{-1}{d_1^2}\frac{\partial d_1}{\partial z} + \frac{1}{d_2^2}\frac{\partial d_2}{\partial z}\right]\Bigg|_{z=0}$$

$$= \frac{Q}{4\pi}\left[\frac{1}{d_1^2}\frac{z-L}{d_1} - \frac{1}{d_2^2}\frac{z+L}{d_2}\right]\Bigg|_{z=0}$$

$$= \frac{Q}{4\pi d^3}(-2L) \qquad d = \sqrt{x^2+y^2+L^2}$$

$$\boxed{\sigma(x,y) = \frac{-QL}{2\pi(x^2+y^2+L^2)^{3/2}}}$$

this complicated density on xy-plane is mimiched by single image charge.

Notice, the total charge on the xy-plane is given by

$$Q_{TOTAL} = \int_0^{2\pi}\int_0^{\infty}\frac{-QL}{2\pi(s^2+L^2)^{3/2}}\, s\,ds\,d\phi$$

$$= \frac{QL}{\sqrt{s^2+L^2}}\Bigg|_0^{\infty}$$

$$= \frac{QL}{\sqrt{\infty}} - \frac{QL}{\sqrt{L^2}}$$

$$= -Q \quad \text{(neat)}$$