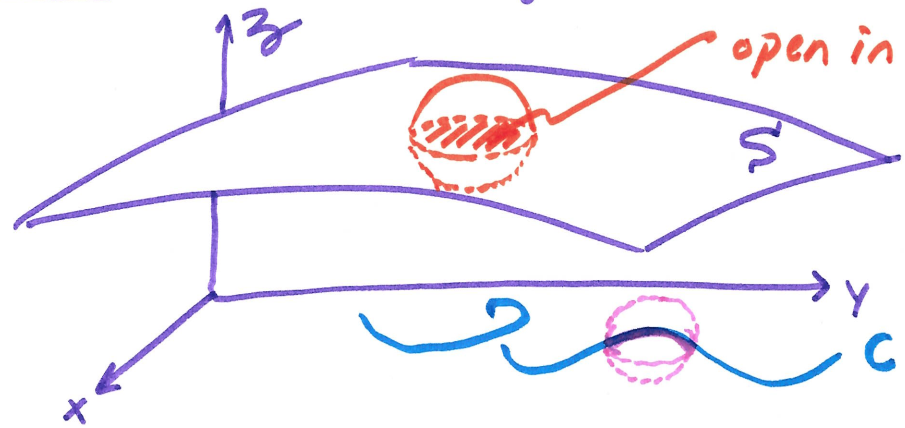
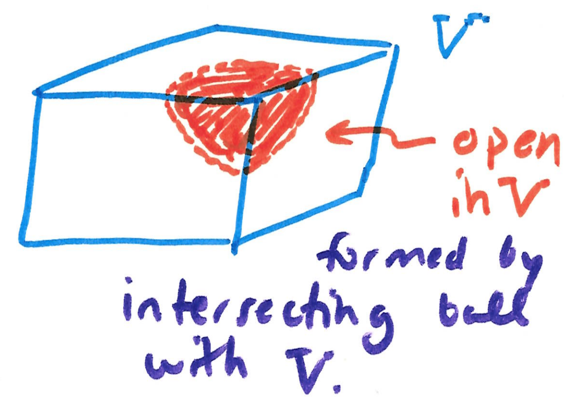
LECTURE 8: SUBSPACES & IMMERSIONS (§3.5 Manetti)

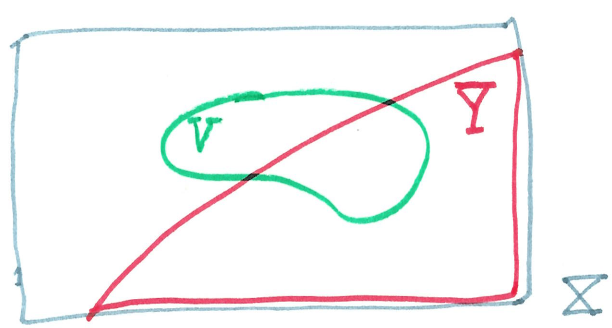
Def^{n}/ Let X be a topological space and $Y \subseteq X$ then the <u>subspace topology</u> on Y is defined by stating $U \subseteq Y$ is open iff $\exists\, V \subseteq X$ open with $U = V \cap Y$.

$\boxed{E1}$ Let $X = \mathbb{R}^3$ with the Euclidean topology then $S \subseteq \mathbb{R}^3$ a <u>surface</u> can be topologized via the subspace topology. Same for volume $V \subseteq \mathbb{R}^3$





Suppose $C \subseteq X$ is closed then $X - C$ is open and we note $Y - (V \cap Y) = Y \cap (X - V)$. Thus,



Th^{m}/ $C \subseteq X$ is closed iff $Y \cap C$ is closed in Y.

$Y \cap C$ closed in Y $\Leftrightarrow$ $Y - (Y \cap C)$ open in Y
$\Leftrightarrow$ $Y \cap (X - C)$ open in Y
$\Leftrightarrow$ $X - C$ open in X
$\Leftrightarrow$ C closed in X. //

Proposition: if $\mathcal{B}$ is topological basis for X then $\{A \cap Y \mid A \in \mathcal{B}\}$ forms topological basis for $Y \subseteq X$ with the subspace topology. Moreover, the subspace top. is a topology

<u>Proof</u>: since $\bigcup_{A \in \beta} A = X$ it is clear: $\bigcup_{A \in \beta} A \cap Y = \left(\bigcup_{A \in \beta} A\right) \cap Y$ (see pg. 24 special case of dist. law

$$= X \cap Y$$
$$= Y.$$

thus $\{A \cap Y \mid A \in \beta\}$ covers Y.

Also, If $A \cap Y$, $B \cap Y$ intersect at $x \in (A \cap Y) \cap (B \cap Y)$ then $x \in (A \cap B) \cap Y \Rightarrow x \in A \cap B \Rightarrow \exists\, C \in \mathcal{B}$ s.t. $x \in C \subseteq A \cap B$ hence $C \cap Y$ is in $\{A \cap Y \mid A \in \mathcal{B}\}$ and $x \in C \cap Y \subseteq (A \cap Y) \cap (B \cap Y)$ all of this to say Th$^{\underline{m}}$ 3.7 applies and we see $\{A \cap Y \mid A \in \mathcal{B}\}$ is a basis for a topology on Y. This topology is the subspace topology.//

Proposition: $i : Y \longrightarrow X$ defined for $Y \subseteq X$ by $i(a) = a \ \forall a \in Y$ is called the <u>inclusion map</u>. The inclusion map is continuous. Moreover the subspace topology on Y is the coarsest topology under which the inclusion is continuous.

<u>Proof</u>: Let $U \subseteq X$ be open then $i^{-1}(U) = U \cap Y$ $\underbrace{\text{which is open}}_{\text{in the subspace topology}}$. Hence i continuous. If $U \cap Y$ was not open in Y then that alternate (non-subspace) topology for Y would have the inclusion map discontinuous.//

$\boxed{E2}$ **Suppose** (X, d) is a metric space and $Y \subseteq X$ then $d|_{Y\times Y} : Y \times Y \to \mathbb{R}$ or $d_Y = d|_{Y\times Y}$ with $d_Y(x,y) = d(x,y)$ $\forall x, y \in Y$ defines (Y, d_Y) a metric space. Moreover, the topology induced on Y by the d_Y-metric is precisely the subspace topology formed from the usual metric topology on X.
(this was illustrated by E1 already)

An open ball in Y comes from $Y \cap (\text{open ball in } X)$.

Def^{n}/ $Y \subseteq X$ is a <u>discrete</u> subspace iff the subspace topology is discrete.
Equivalently, $Z \subseteq X$ is <u>discrete</u> iff for any $z \in Z$ there exists open $U \subseteq X$ s.t. $U \cap Z = \{z\}$.

$\boxed{E3}$ $\mathbb{Z} \subseteq \mathbb{R}$ is discrete, but $\mathbb{Q} \subseteq \mathbb{R}$ is not discrete.

$\boxed{E4}$ any subset of X with the discrete topology $\mathcal{T} = \mathcal{P}(X)$ is discrete. Every subspace of the discrete topology is discrete.

Proposition (3.54)

Let X, Z be topological spaces, Y a subspace of X and $f : Z \longrightarrow Y$ a map and $i \circ f : Z \longrightarrow X$. If $i \circ f$ is continuous then also f is continuous, and conversely.

Proof: $\Rightarrow$] Suppose $i \circ f$ is continuous. Let $V \subseteq Y$ then $i(V) = V$ since $i(x) = x$ $\forall x \in Y$ where $i : Y \longrightarrow X$ is the inclusion map. then $f^{-1}(V) = f^{-1}(i(V))$ but $i(V) = V$ also yields $i^{-1}(V) = V$

hence $f^{-1}(V) = f^{-1}(i^{-1}(V))$

$= (i \circ f)^{-1}(V)$ which is open by cont. of $i \circ f$

$\therefore$ f is continuous.//

$\Leftarrow$] If f is continuous then let $U \subseteq X$ be open and note $(i \circ f)^{-1}(U) = f^{-1}(\underbrace{i^{-1}(U)}_{\text{open in subspace } Y})$ = open by cont. of f

Thus $i \circ f$ is continuous.//

Lemma (3.55):

Let Y be subspace of X a topological space and suppose A is a subset of Y. The closure of A in Y coincides with the intersection of Y with the closure of A in X

Proof: let $\mathcal{C}$ be the family of closed sets in X which contain A.

But, we know the only closed subsets of Y are those which coincide with the intersection of a closed set in X with Y.

$$\underset{\text{closed}}{C'} \subseteq Y \iff C' = Y \cap C \quad \text{where } C \text{ closed in } X$$

If $\mathcal{C}'$ is the family of closed sets in Y which contain A then

$$\underbrace{\bigcap_{\substack{C' \in \mathcal{C}' \\ A \subseteq C'}} C'}_{\text{closure of } A \text{ in } Y} = \bigcap_{\substack{C \in \mathcal{C} \\ A \subseteq C}} Y \cap C = \Big(\bigcap_{\substack{C \in \mathcal{C} \\ A \subseteq C}} C\Big) \cap Y = \underbrace{\bar{A}}_{\text{closure of } A \text{ in } X} \cap\, Y$$

Lemma (3.56):

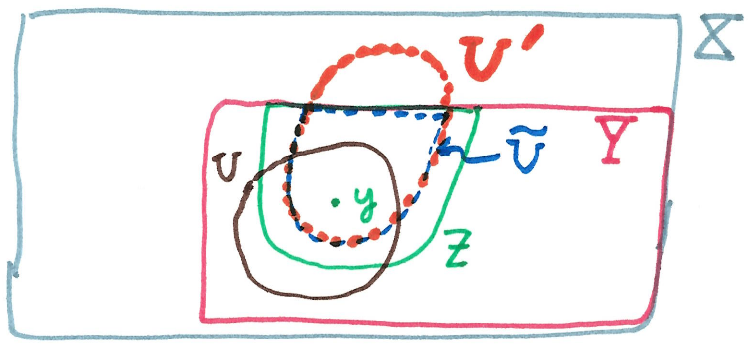
Let X be topological space and $Z \subseteq Y \subseteq X$

(1.) If Y open in X then Z open in Y iff Z open in X

(2.) If Y closed in X then Z closed in Y iff Z closed in X

(3.) If Y is nbhd. of y, Z is nbhd. of y in Y iff Z is nbhd of y in X.

Proof: (1.) and (2.) left to reader. Look at (3.), suppose Y is nbhd of y. then $\exists\, U$ open in X containing y with $y \in U \subseteq Y$. Suppose Z is nbhd. of y in Y then $\exists\, \tilde{U} \subseteq Z$ with $\tilde{U}$ open in Y and $y \in \tilde{U} \subseteq Z$ and since $\tilde{U}$ open in Y as subspace $\Rightarrow \exists\, U'$ open in X such that $\tilde{U} = Y \cap U'$



Then $y \in U \cap U' \subseteq \tilde{U} \subseteq Z$

and $U \cap U'$ is open in X

$\therefore$ Z is nbhd of y in X.

(I leave the converse direction of:

Z nbhd of y in $X \Rightarrow Z$ nbhd. of y in Y

to the reader.)

Def^{n}/ A <u>continuous</u>, one-to-one map $f: X \to Y$ is called an <u>immersion</u> if every open set in X is of the form $f^{-1}(A)$ for some open set A in Y.

In other words, $f: X \to Y$ is a topological immersion iff it induces $\hat{f}: X \to f(X)$ a homeomorphism. $\hat{f}(x) = f(x)\ \forall x \in X$ ← just changing codomain from Y to $f(X)$ to make $\hat{f}$ surjective.

<u>Remark</u>: not all continuous 1-1 maps are topological immersions. For instance,

$$Id : (\mathbb{R}, \text{Euclidean Top.}) \longrightarrow (\mathbb{R}, \underbrace{\text{trivial top}}_{\mathcal{T} = \{\emptyset, \mathbb{R}\}})$$

$$Id^{-1}(\emptyset) = \emptyset$$
$$Id^{-1}(\mathbb{R}) = \mathbb{R}$$

many open sets in Euclidean Top. for $\mathbb{R}$ besides $\emptyset$ and $\mathbb{R}$!

Def^{n}/ A <u>closed</u> <u>immersion</u> is both an immersion and a closed map. Likewise an <u>open</u> <u>immersion</u> is both an immersion and an open map.

<u>Lemma (3.59)</u>: Let $f: X \to Y$ be continuous.
(1.) If f is closed and 1-1 then f is closed immersion.
(2.) If f is open and injective then f is an open immersion.

<u>Proof</u>: Suppose f is 1-1, continuous and closed. If $C \subseteq X$ closed then $f(C)$ closed subset $f(C) \subseteq f(X) \subseteq Y$ thus $f(C)$ closed in $f(X)$ and hence $f: X \to f(X)$ is cont, bijective and closed. //