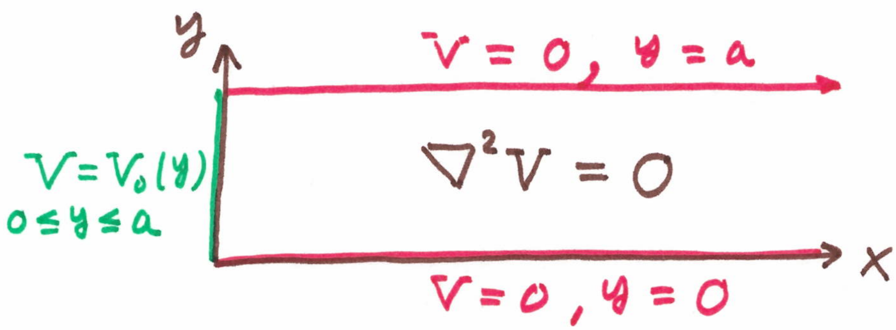
LECTURE 9: SOLVING LAPLACE'S EQ$^{\underline{n}}$: CARTESIAN CASE

In the absence of charge Poisson's Eq: $\nabla^2 V = -\rho/\varepsilon_0$ simplifies to Laplace's Eq: $\nabla^2 V = 0$. It turns out there are elegant solutions for such equation. In this talk we examine how separation of variables and the Fourier technique allow solution to Cartesian problems.

E1 Given two conducting half-planes, both at $V=0$ parallel to the xz-plane at $y=0$ and $y=a$, then another voltage is applied at $x=0$ of the form $V=V_0(y)$. Finally $V \to 0$ as $x \to \infty$



$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

By symmetry, no z-dependence, so we drop z until answer.

$$\boxed{V_{xx} + V_{yy} = 0} \; - *$$

Ansatz: $V(x,y) = X(x)\,Y(y)$

$$V_{xx} = X''(x)\,Y(y) = X''Y$$

$$V_{yy} = X(x)\,Y''(y) = XY''$$

Plugging guess into $*$, $X''Y + XY'' = 0$

Continuing,

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} + \frac{Y''}{Y} = 0 \hookrightarrow \frac{X''}{X} = \frac{-Y''}{Y} = \Upsilon$$

Remark: $\partial_y\left(\frac{X''}{X}\right) = 0$ and $\partial_x\left(\frac{-Y''}{Y}\right) = 0$ hence Υ must be a constant.

Boundary Conditions

BC1: $V(x, a) = X(x)\,Y(a) = 0$

BC2: $V(x, 0) = X(x)\,Y(0) = 0$

$\Big\}\; Y(a) = Y(0) = 0$

BC3: $V(0, y) = X(0)\,Y(y) = V_0(y)$

BC4: $V(\infty, y) = 0$

We wish to solve $X'' - \Upsilon X = 0$ and $Y'' + \Upsilon Y = 0$ subject the given BC's. We focus on BC1/BC2 to begin. Notice Υ has 3 cases,

1.) $\Upsilon = 0$, $Y'' = 0 \;\therefore\; Y(y) = \bar{a} + y\bar{b}$

and $Y(0) = \bar{a} = 0$ and $Y(a) = a\bar{b} = 0 \;\therefore\; \underline{Y = 0}$

2.) $\Upsilon = -\beta^2$, $Y'' - \beta^2 Y = 0$

$\therefore\; Y = c_1 \cosh\beta y + c_2 \sinh\beta y$

$Y(0) = c_1 = 0$

$Y(a) = c_2 \sinh(\beta a) = 0 \Rightarrow c_2 = 0$

$\therefore\; \underline{Y(y) = 0}$

3.) $\Upsilon = \beta^2$, $Y'' + \beta^2 Y = 0$

$$Y(y) = c_1 \cos(\beta y) + c_2 \sin(\beta y)$$

$$Y(0) = c_1 = 0$$

$$Y(a) = c_2 \underbrace{\sin(\beta a)}_{\beta a = n\pi \text{ for } n \in \mathbb{N}} = 0$$

$$\beta = \frac{n\pi}{a} \longrightarrow \Upsilon = \frac{n^2\pi^2}{a^2}$$

$$\boxed{Y_n(y) = \sin\left(\frac{n\pi y}{a}\right)}$$

Continuing, now we face $X'' - \Upsilon X = 0$ subject $\Upsilon = \frac{n^2\pi^2}{a^2}$ thus $X'' - \frac{n^2\pi^2}{a^2} X = 0$

$$X(x) = c_1 \exp\left(\frac{n\pi x}{a}\right) + c_2 \exp\left(\frac{-n\pi x}{a}\right)$$

Remark: alternatively $X(x) = b_1 \cosh\left(\frac{n\pi x}{a}\right) + b_2 \sinh\left(\frac{n\pi x}{a}\right)$ however BC4 is better implemented with the exponential formulation in this case.

BC4 $V(\infty, y) = X(\infty) Y(y) = 0 \Rightarrow c_1 = 0$.

$$\boxed{X_n(x) = \exp\left(\frac{-n\pi x}{a}\right)}$$

Next we use the characteristic solutions $X_n Y_n$ to fit BC3, however, we use $\sum_{n=1}^{\infty} X_n Y_n$ in order to capture the particular form given in BC3 ↩

Continuing,

$$V(x,y) = \sum_{n=1}^{\infty} A_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$$

Apply BC3,

$$\boxed{V(0,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{a}\right) = V_0(y)} \quad (**)$$

Remark: we want to find formular for A_n in $(**)$ however this requires a bit of calculus we may not know... so let's work it out,

$$\sin(\theta)\sin(\beta) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})\frac{1}{2i}(e^{i\beta} - e^{-i\beta})$$

$$= \frac{-1}{4}\left(e^{i(\theta+\beta)} + e^{-i(\theta+\beta)} - e^{i(\theta-\beta)} - e^{-i(\theta-\beta)}\right)$$

$$= \frac{1}{2}\left[\frac{1}{2}\left(e^{i(\theta-\beta)} + e^{-i(\theta-\beta)}\right) - \frac{1}{2}\left(e^{i(\theta+\beta)} + e^{-i(\theta+\beta)}\right)\right]$$

$$= \frac{1}{2}\left[\cos(\theta-\beta) - \cos(\theta+\beta)\right]$$

1. $$\int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{k\pi y}{a}\right)dy = \int_0^a \frac{1}{2}\left[\cos\left(\frac{(n-k)\pi y}{a}\right) - \cos\left(\frac{(n+k)\pi y}{a}\right)\right]dy$$

$$= \frac{a}{2\pi(n-k)}\sin(n-k)\pi - \frac{a}{2\pi(n+k)}\sin(n+k)\pi$$

$$= 0 \quad (\text{provided } n \neq \pm k)$$

2. $$\int_0^a \sin^2\left(\frac{n\pi y}{a}\right)dy = \int_0^a \frac{1}{2}\left(1 - \cos\left(\frac{2n\pi y}{a}\right)\right)dy = \frac{a}{2}$$

$$\boxed{\underline{\text{Proposition}}: \int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{k\pi y}{a}\right)dy = \frac{a}{2}\delta_{n,k}}$$

Continuing, multiply ** by $\sin\left(\frac{k\pi y}{a}\right)$ and integrate from $y=0$ to $y=a$,

$$\int_0^a V_0(y)\sin\left(\frac{k\pi y}{a}\right)dy = \int_0^a\left(\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{a}\right)\right)\sin\left(\frac{k\pi y}{a}\right)dy$$

$$= \sum_{n=1}^{\infty} A_n \int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{k\pi y}{a}\right)dy$$

by Proposition

$$= \sum_{n=1}^{\infty} A_n \frac{a}{2}\delta_{n,k}$$

$$= \frac{a A_k}{2} \quad \therefore \quad \underline{A_k = \frac{2}{a}\int_0^a V_0(y)\sin\left(\frac{k\pi y}{a}\right)dy}.$$

In summary,

$$\boxed{\begin{aligned} V(x,y,z) &= \sum_{n=1}^{\infty} A_n e^{-\frac{n\pi x}{a}}\sin\left(\frac{n\pi y}{a}\right) \\ \text{where } A_n &= \frac{2}{a}\int_0^a V_0(y)\sin\left(\frac{n\pi y}{a}\right)dy \end{aligned}}$$

$\boxed{E2}$ Same BC's as E1, but suppose

$$V_0(y) = \underbrace{10\sin\left(\frac{3\pi y}{a}\right) - 40\sin\left(\frac{7y\pi}{a}\right) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{a}\right)}_{\text{read off } A_3 = 10,\ A_7 = -40}$$

$A_n = 0$ for $n \neq 3, 7$

$$\boxed{V(x,y,z) = 10e^{-\frac{3\pi x}{a}}\sin\left(\frac{3\pi y}{a}\right) - 40e^{-\frac{7\pi x}{a}}\sin\left(\frac{7\pi y}{a}\right)}$$

$\boxed{E3}$ once more, same BC's as $\boxed{E1}$ but with $\underbrace{V_0(y) = V_0}_{\text{constant}}$

We must solve the Fourier integral, the coefficient A_n is discussed in the larger program of Fourier analysis (we just use for our Physics, a broader discussion of the technique and convergence properties of Fourier series we leave for a math course)

$$A_n = \frac{2}{a}\int_0^a V_0 \sin\left(\frac{n\pi y}{a}\right) dy$$

$$= \frac{2V_0}{a}\left[\frac{-a}{n\pi}\cos\left(\frac{n\pi y}{a}\right)\right]\Bigg|_0^a$$

$$= \frac{-2V_0}{n\pi}\left[\cos(n\pi) - 1\right]$$

$$= \frac{2V_0}{n\pi}\left[1 - (-1)^n\right]$$

Remark: thinking is hard some days.

$$= \begin{cases} 0 & : n \text{ even} \\ \dfrac{4V_0}{n\pi} & : n \text{ odd} \end{cases}$$

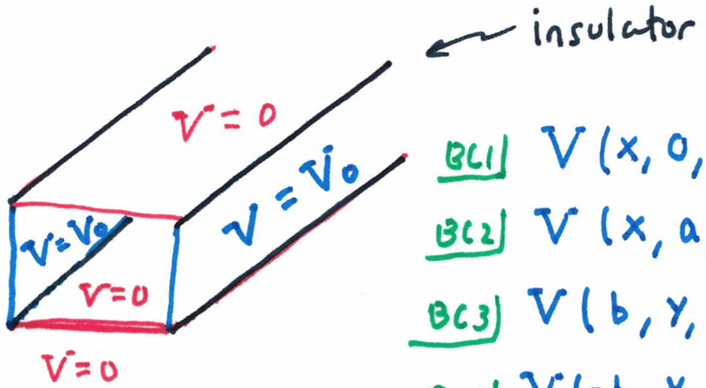
$$= \begin{cases} 0 & : n = 2k, \ k \in \mathbb{N} \\ \dfrac{4V_0}{(2k-1)\pi} & n = 2k-1, \ k \in \mathbb{N} \end{cases}$$

Thus $A_{2k-1} = \dfrac{4V_0}{(2k-1)\pi}$ for $k \in \mathbb{N}$ and $A_{2k} = 0$. Hence,

$$\boxed{V(x,y,z) = \sum_{k=1}^{\infty} \frac{4V_0}{(2k-1)\pi} \exp\left(\frac{-(2k-1)\pi x}{a}\right) \sin\left(\frac{(2k-1)\pi y}{a}\right)}$$

(see Griffith's Figure 3.18 on p. 133 of 5^{th} Edition for a plot of this potential in a sense)

$\boxed{E4}$ Given two metal plates at $y=0$ and $y=a$ both with $V=0$ are connected at $x=\pm b$ by metal strips at $V=V_0$. Find potential in this conducting pipe (the corners are insulated)



$\underline{BC1}$ $V(x,0,z)=0$
$\underline{BC2}$ $V(x,a,z)=0$
$\underline{BC3}$ $V(b,y,z)=V_0$
$\underline{BC4}$ $V(-b,y,z)=V_0$

given boundary condition.

Once more we drop z since it is clear by symmetry there is no non-constant function of z for V;

$$\boxed{V(x,y,z) = V(x,y)}$$

Once more $V(x,y) = X(x)Y(y)$ and this time,

$\underline{BC1}$ $Y(0)=0$
$\underline{BC2}$ $Y(a)=0$

$$\frac{X''}{X} = \frac{-Y''}{Y} = \Upsilon$$

Is the same as it was for $\boxed{E1}$ hence $\Upsilon = \frac{n^2\pi^2}{a^2}$ and $Y_n(y) = \sin\left(\frac{n\pi y}{a}\right)$. Likewise $X'' - \frac{n^2\pi^2}{a^2}X = 0$ has solutions of the form

$$X_n(x) = A_n\cosh\left(\frac{n\pi x}{a}\right) + B_n\sinh\left(\frac{n\pi x}{a}\right)$$

Hence, we form potential potential by infinite sum of such product solutions $\underbrace{X_n(x)\,Y_n(y)}_{\text{solve BC1 and BC2}}$

(next BC3 and BC4 choose A_n, B_n)

E4 continued (the potential potential)

$$\boxed{V(x,y) = \sum_{n=1}^{\infty} \left[A_n \cosh\left(\frac{n\pi x}{a}\right) + B_n \sinh\left(\frac{n\pi x}{a}\right) \right] \sin\left(\frac{n\pi y}{a}\right)}$$

Recall $V(b,y) = V_o = V(-b,y)$ for all y. This means the solution should be even in x thus $B_n = 0$ and we reduce our potential potential to,

$$\boxed{V(x,y) = \sum_{n=1}^{\infty} A_n \cosh\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right)}$$

Once more we use Fourier's trick, consider $V(b,y) = V_o$,

$$V_o = \sum_{n=1}^{\infty} A_n \cosh\left(\frac{n\pi b}{a}\right) \sin\left(\frac{n\pi y}{a}\right)$$

$$\int_0^a V_o \sin\left(\frac{n\pi k}{a}\right) dy = \int_0^a \left(\sum_{n=1}^{\infty} \cosh\left(\frac{n\pi b}{a}\right) A_n \sin\left(\frac{n\pi y}{a}\right) \right) \sin\left(\frac{k\pi y}{a}\right) dy$$

$$= \sum_{n=1}^{\infty} A_n \cosh\left(\frac{n\pi b}{a}\right) \underbrace{\int_0^a \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{k\pi y}{a}\right) dy}_{\frac{a}{2}\delta_{n,k}}$$

$$= \frac{a A_k \cosh\left(\frac{k\pi b}{a}\right)}{2}$$

$$\therefore \quad A_k = \frac{2}{a\cosh\left(\frac{k\pi b}{a}\right)} \int_0^a V_o \sin\left(\frac{k\pi y}{a}\right) dy$$

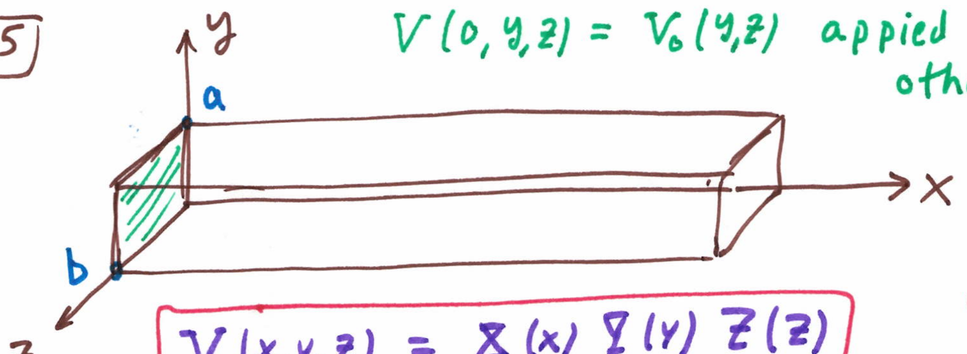
I prefer n,

$$A_n = \frac{2V_o}{a\cosh\left(\frac{n\pi b}{a}\right)} \cdot \frac{a}{n\pi}\left[-\cos(n\pi) + 1\right] = \begin{cases} 0 & : n \text{ even} \\ \dfrac{4V_o}{n\pi\cosh\left(\frac{n\pi b}{a}\right)} & : n \text{ odd} \end{cases}$$

E4 continued $A_{2k-1} = \dfrac{4V_o}{(2k-1)\pi \cosh\left(\frac{(2k-1)\pi b}{a}\right)}$

$$\boxed{V(x,y,z) = \sum_{k=1}^{\infty} A_{2k-1} \cosh\left(\frac{(2k-1)\pi x}{a}\right) \sin\left(\frac{(2k-1)\pi y}{a}\right) \quad \text{where} \quad A_{2k-1} = \frac{4\,V_o}{(2k-1)\pi \cosh\left(\frac{(2k-1)\pi b}{a}\right)}}$$

E5



$V(0,y,z) = V_o(y,z)$ appied to end of tube
other sides set
to $V = 0$

$$\boxed{V(x,y,z) = X(x)\,Y(y)\,Z(z)}$$

can't suppress z in this problem.

Essentially same approach as E1 applies here from *

$$\partial_x^2 V + \partial_y^2 V + \partial_z^2 V = 0$$

$$X'' Y Z + X Y'' Z + X Y Z'' = 0$$

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$

Analysis of given BC's for y and z suggest we set

$$\frac{X''}{X} = -\frac{Y''}{Y} - \frac{Z''}{Z} = k^2 + l^2$$

logic

where $\underbrace{Y'' + k^2 Y = 0}_{Y(0) = Y(a) = 0}$ I and $\underbrace{Z'' + l^2 Z = 0}_{Z(0) = Z(b)}$ II

$$k = \frac{m\pi}{a} \qquad l = \frac{m\pi}{b}$$

ES continued

$$\frac{X''}{X} = k^2 + l^2 \longrightarrow X'' - (k^2+l^2)X = 0 \quad \text{III}$$

Solve I, II and III to obtain,

$$X(x) = A\exp\left(x\sqrt{k^2+l^2}\right) + B\exp\left(-x\sqrt{k^2+l^2}\right)$$

$$Y(y) = C\sin(ky) + D\cos(ky) \qquad k = \frac{n\pi}{a}$$

$$Z(z) = E\sin(lz) + F\cos(lz) \qquad l = \frac{m\pi}{b}$$

Boundary conditions force $D = 0$, $F = 0$, $\underbrace{A = 0}_{V(\infty, y, z) = 0}$

Therefore the potential potential is constructed,

$$\boxed{V(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m}\, e^{-\pi x\sqrt{n^2/a^2 + m^2/b^2}} \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi z}{b}\right)}$$

Once more the Fourier trick is used to solve for $C_{m,n}$ in terms of an appropriate integration,

$$\boxed{C_{n,m} = \frac{4}{ab}\int_0^a\int_0^b V_0(y,z)\sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi z}{b}\right)dz\,dy}$$

Griffiths has solution for case $V_0(y,z) = V_0$ on p. 138.