LECTURE 9 : TOPOLOGICAL PRODUCTS (§3.6 Manetti)

Let P, Q be topological spaces then the projection maps on $P \times Q$ are $\pi_1 : P \times Q \longrightarrow P$ and $\pi_2 : P \times Q \longrightarrow Q$ are defined by $\pi_1(x,y) = x$ and $\pi_2(x,y) = y$ $\forall (x,y) \in P \times Q$.

Def^n/ The <u>product</u> <u>topology</u> on $P \times Q$ is the coarsest topology amongst those topologies which make π_1, π_2 continuous.

Th^m/ (1.) subsets $U \times V$ where U is open in P and V is open in Q form a basis known as the <u>canonical</u> <u>basis</u> of the product topology.

(2.) the projections π_1 & π_2 are open maps and for any $(x,y) \in P \times Q$ the restrictions $\pi_1 : P \times \{y\} \longrightarrow P$ and $\pi_2 : \{x\} \times Q \longrightarrow Q$ are homeomorphisms.

(3.) a map $f : X \longrightarrow P \times Q$ is continuous iff $f_1 = \pi_1 \circ f$, $f_2 = \pi_2 \circ f$ are continuous.

<u>Proof</u>: (1.) let P be the product topology. Notice the collection of subsets $U \times V$ where U open in P and V open in Q satisfy the conditions of Th^m 3.7 since $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$ hence such subsets form basis for topology $\mathcal{T}$ for $P \times Q$. Notice $\pi_1^{-1}(U) = U \times Q$ and $\pi_2^{-1}(V) = P \times V$ for any open U, V thus π_1 & π_2 are continuous w.r.t $\mathcal{T}$ $\Rightarrow$ P coarser than $\mathcal{T}$. However, also note $U \subseteq P$, $V \subseteq Q$ open give $U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V) \in P$ $\Rightarrow$ any open set of $\mathcal{T}$ is union of sets in product topology $\Rightarrow$ $\mathcal{T}$ coarser than P $\therefore$ $\mathcal{T} = P$.

Proof continued

(2.) Observe $(U \times V) \cap (P \times \{y\}) = \begin{cases} U \times \{y\} & \text{if } y \in V \\ \emptyset & \text{if } y \notin V \end{cases}$

Likewise $(U \times V) \cap (\{x\} \times Q) = \begin{cases} \{x\} \times V & \text{if } x \in U \\ \emptyset & \text{if } x \notin U \end{cases}$

this allows us to understand the subspace topologies of $P \times \{y\}$ and $\{x\} \times Q$

Notice $\pi_1 : P \times \{y\} \to P$ and $\pi_2 : \{x\} \times Q \to Q$ are clearly bijections and if $U \subseteq P$ is open then $\pi_1^{-1}(U) = U \times \{y\}$ is open $\therefore \pi_1$ continuous. Likewise for π_2. Observe also, $\pi_1(\emptyset) = \emptyset$ and $\pi_2(\emptyset) = \emptyset$ whereas $\pi_1(U \times \{y\}) = U$ and $\pi_2(\{x\} \times V) = V$ and it follows that π_1, π_2 are open maps (technically these are restrictions of π_1, π_2)

(3.) Consider $f : X \to P \times Q$ continuous, then $f_1 = \pi_1 \circ f$, $f_2 = \pi_2 \circ f$ are continuous since the composition of continuous maps is continuous. Conversely, if $f_1 = \pi_1 \circ f$ and $f_2 = \pi_2 \circ f$ are continuous then for $U \times V$ open in $P \times Q$ notice

$$\begin{aligned}
f^{-1}(U \times V) &= f_1^{-1}(U) \cap f_2^{-1}(V) \\
&= (\pi_1 \circ f)^{-1}(U) \cap (\pi_2 \circ f)^{-1}(V) \\
&= [(f^{-1} \circ \pi_1^{-1})(U)] \cap [f^{-1} \circ \pi_2^{-1}(V)] \\
&= f^{-1}(U \times Q) \cap f^{-1}(P \times V) \\
&= \{x \in X \mid f(x) \in U \times Q\} \cap \{x \in X \mid f(x) \in P \times V\} \\
&= \{x \in X \mid f(x) \in U \times V\}.
\end{aligned}$$

Def

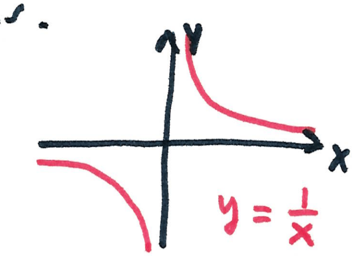
Def^n/ The product topology on $P_1 \times P_2 \times \cdots \times P_n$ where $P_1, P_2, \ldots, P_n$ are topological spaces is the coarsest among those which make $\pi_1, \pi_2, \ldots, \pi_n$ continuous. Moreover, the cannonical basis of the product topology is given by $U_1 \times \cdots \times U_n$ where U_i are open in P_i $\forall i = 1, 2, \ldots, n$.

$\boxed{E1}$ The Euclidean topology for $\mathbb{R}^n$ coincides with the product topology for n-copies of $\mathbb{R}$; $\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n\text{- fold copies}}$

$\boxed{E2}$ The unrestricted $\pi_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $\pi_1(x,y) = x$ is not a closed map. (why?) HERE'S WHY

Consider the hyperbola $\mathcal{H} = \{(x,y) \mid xy = 1\} \subseteq \mathbb{R}^2$ is the zero set of $f(x,y) = xy - 1$ and f is continuous. Note $\mathcal{H} = f^{-1}\{0\}$ $\therefore$ $\mathcal{H}$ is closed subset of $\mathbb{R}^2$

Yet, $\pi_1(\mathcal{H}) = (-\infty, 0) \cup (0, \infty)$ is open.



Remark: I am not a fan of Manetti's lack of notation for π_1 restricted vs. π_1 unrestricted. There should be some notation used as to help avoid confusion.